

## The Four Colour Theorem: A Proof by Induction

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### Abstract

A simple proof of the **Four Colour Theorem** which states that four colours are sufficient to colour a *map*, has been presented in this article. The proof is not algorithm dependent. The theorem was on colouring of maps while our proof is directly with reference to maps. We have observed that when a map is depicted as a graph, the graph does not show certain things which are visible only in the maps concerned. Indeed for a number of maps there can be just one single graph. Therefore, what can immediately be observed in a map may not be observable in the graph concerned. We have first shown that if a map is of exactly four regions every region connected to every other, at least one of the regions cannot have any common boundary with a fifth region outside the map. Using this assertion, we have proved that three colours are sufficient to colour the boundary regions of a map. Finally, we have proved by induction that four colours are sufficient to colour a map.

*Key Words: Map colouring, graph theory, planar map, FCT*

### 1. Introduction:

In graph theory, a *graph* is made up of nodes that are connected by edges. In a *map*, the regions can be represented as nodes and every pair of connected regions can be represented by two nodes and an edge joining the two nodes. If two regions are connected, the two nodes representing the regions would have an edge joining the nodes. In this article, we are going to show that connectivity of the regions in a map can describe some details that remain hidden if the map is represented as a *graph*.

The **Four Colour Theorem** which is popularly known as **FCT**, states that four colours are sufficient to colour a map in such a way that any two connected regions will have two different colours. This particular property of a map was conjectured way back in 1852. It remained a conjecture for one and a quarter of a century. Finally in 1977, the first part of a computer dependent proof appeared [Appel and Haken (1977)]. The second part of the proof appeared in the same year [Appel, Haken and & Koch (1977)]. We were interested to see whether there can be a proof outside the theory of graphs. We were guided by an idea that such a simple looking theorem, to understand the statement of which it does not require any knowledge of mathematics, should have a proof based directly on maps.

People were trying to prove the conjecture from graph theoretic standpoint, which is why the first ever proof was a computer dependent one. We are going to put forward a proof based directly on maps. Our approach is a different one.

Some of the matters included in this article were published in 2010 [Baruah (2010)]. We have dropped one of the four theorems included in that article and have completely changed the proofs of two other theorems. We have added discussions on clusters of three connected regions in any given map, and have made the proof of the main theorem a refined one.

## **2. Views on Map Colouring:**

Let us now proceed to explain our perspective step by step. When we consider a map with just two regions connected to each other, there is a possibility that one of the regions is enveloped by the other. If we consider a map with three regions with every region connected to every other, there is a possibility that one or two of the three regions may be enveloped. However, if we consider a map with four regions with every region connected to every other, it is sure that at least one of the regions will be enveloped. This fact with reference to a map with four interconnected regions leads to a simple proof of the **FCT**.

Assume that two nodes A and B in a graph are connected by an edge. Now if we represent the graph from the perspective of two regions of a map, we can easily see that there are two different ways in which that can happen:

- (1) No region is enveloped.
- (2) One of the regions envelopes the other region.

But if we represent these different cases mentioned above with the help of a graph, there can be only one single graph. With the help of this simple example, we can see that if we look into the matter of connectivity of regions, a graph cannot show certain things that can be viewed with the help of a map.

In a map, if there are three regions A, B, C, every region connected to every other, the following are the possibilities:

- (1) No region is enveloped,
- (2) One region is enveloped by the other two regions, and
- (3) Two regions are enveloped by the third region.

It can be easily seen that in all these cases, three colours will be sufficient to colour the map.

$${}^3C_0 + {}^3C_1 + {}^3C_2 = 7$$

There can be different combinations, although in the form of a graph there can be only one graph. The graph will depict that all three nodes can have an edge with a fourth node. The maps however will depict that a maximum of two regions may not have any common boundary with a region outside the map.

Now let us see what happens when in a map there are four regions every region connected to every other. Consider **Figure 1** shown below. Regions A, B, C and D are connected to one another. In Graph theory, this graph is known as  $K_4$ . It is a planar graph in the sense that a fifth node E cannot have an edge connecting it with node A without crossing the edges BC, BD and DC; an edge AE cannot be on the same plane on which the graph has been drawn. There can be

$$4 = {}^4C_1$$

such cases in which one of the regions gets enveloped.

Next, consider **Figure 2**, in which regions A and B are enveloped by the other two regions. Note that we end up getting the same graph which was shown in **Figure 1**. As far as the map is concerned, there can be such cases.

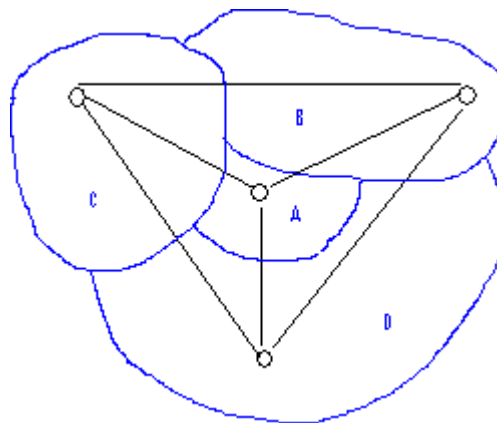
$$6 = {}^4C_2$$

Finally, consider **Figure 3**, in which regions A, B and C have been enveloped by region D. There can be such cases.

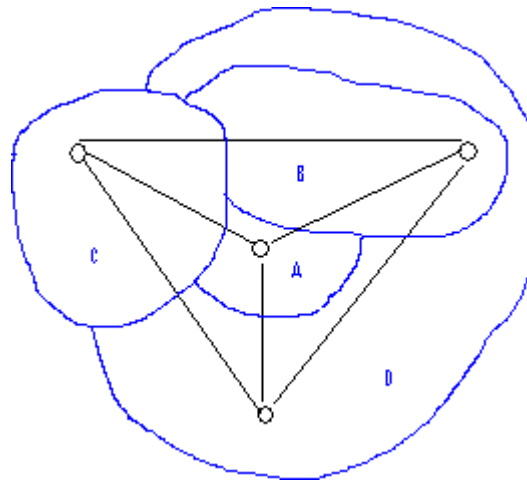
$$4 = {}^4C_3$$

Note that we still have the same graph as in **Figure 1** and **Figure 2**. Other than the cases described in the figures (**Figure 1**, **Figure 2** & **Figure 3**), there cannot be any more possibilities in which there are four regions connected to one another. We have thus demonstrated that if in a map there are four regions such that every region is connected to every other, then at least one of the regions would get enveloped. Note that as soon as a region gets enveloped, it will not have any common boundary with a fifth region outside the map with four regions.

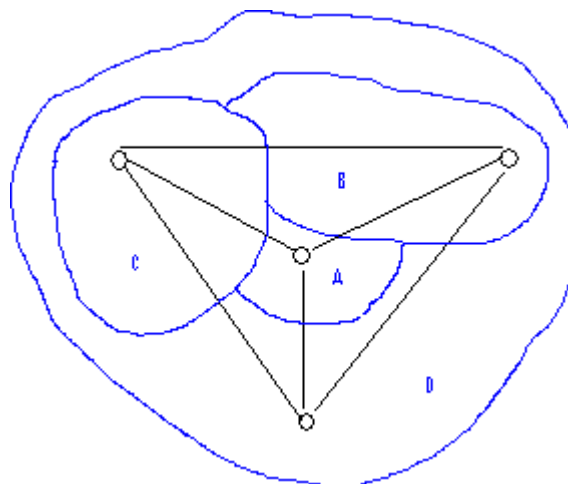
As the number of possibilities in total is finite (= 14), this demonstration is a proof of the following statement presented in **Theorem (1)**:



**Figure 1: One Region Enveloped**



**Figure 2: Two Regions Enveloped**



**Figure 3: Three Regions Enveloped**

**Theorem (1):** In a map with four regions, if every region is connected to the other three, then at least one of the regions would get enveloped.

Theorem (1) leads to the following statement:

**Corollary (1):** In a map with more than four regions, it is not possible that every region is connected to every other.

Now let us proceed to find how many colours would be sufficient to colour the *boundary regions* of a map. As per **Theorem (1)** not all four of four interconnected regions can be on the boundary because at least one of them, being enveloped, cannot be on the boundary of the map. Therefore on the boundary of a map there cannot be a cluster of four regions, every region connected to every other. Hence in the boundary regions of a map there can be clusters of at most three regions only, such that every region in every such cluster is connected to every other region of that cluster.

We would like to explain the matter step by step as follows:

(1) We start with the simplest type of case in which the regions on the boundary are such that one region is connected to a second region which is connected to a third region, and so on, creating a chain. In such a situation it is obvious that no more than three colours would be necessary to colour the boundary regions of a map.

(2) On the boundary of a map, if there happens to be a cluster of three regions in which at least one region is enveloped, then the enveloped region or regions cannot be on the boundary, and therefore in such a cluster of three regions, the regions touching the boundary of the map, will never need more than three colours.

(3) On the boundary, there can be clusters of three regions in which no region is enveloped, *with every region free to be connected to a fourth region*. We have to note that every region of such clusters of three regions must necessarily be a part of the boundary. If there is just one single cluster of three such boundary regions connected to one another, the cluster can be coloured with three colours, and for the other boundary regions of the map the same three colours will be sufficient in the sense that the other regions on the boundary would have to be such that one region from the cluster is connected to a fourth region which is connected to a fifth region, and so on. In such an arrangement, the question of requirement of a fourth colour does not arise.

(4) If in the boundary regions there are more than one clusters of three regions which are *connected to one another*, three colours can be used to colour one of the clusters first. Assume that interconnected regions A, B and C constitute a cluster of three regions on the boundary of a map. Then a fourth boundary region D cannot be connected to all three of A, B and C, because if

that happens then one of the regions A, B and C will be enveloped, and the enveloped region cannot be on the boundary defying thereby our assumption that A, B, C are on the boundary. Therefore, D may be connected to at most two of A, B and which means that D can be coloured with the colour allotted to the region not connected to D, with all four regions being on the boundary. Hence for the second cluster the earlier three colours will again be sufficient. Suppose the second cluster is made up of regions A, B and D. Then a third cluster of three regions A, D and E, for example, may be there with all five regions on the boundary. Thus the same three colours will be sufficient for this third cluster also. This process may continue adding one more new region to two of the earlier regions with all regions remaining on the boundary, and therefore the same three colours would continue to be sufficient for colouring every such new cluster.

(5) Suppose D is connected to just one of A, B and C mentioned above. For example, let D be connected to A, and another region E be connected to A and D. For this second cluster also the same three colours would be sufficient because the colours allotted to B and C can be used to colour regions D and E.

(6) If there are two or more clusters of three interconnected regions such that they have no common region, the same three colours would anyway be sufficient.

(7) The types of possibilities mentioned from (1) to (6) above are exhaustive. There can be no more *types* of possibilities on the boundary of a map. There may be different combinations of the types mentioned herein, all requiring no more than three colours.

Thus we can state the following theorem:

**Theorem (2):** Three colours are sufficient to colour the boundary regions of a map.

It is known that the method of induction of proving a statement involving a positive integer  $n$  is as follows. Assume that the statement is true for  $n = m$ . Verify whether this assumption leads to the conclusion that the statement is true for  $n = m + 1$ . Suppose the statement is found to be true for  $n = m + 1$ . Now if the statement is found to be true for  $n = 1$ , it will then have to be true for  $n = 1 + 1 = 2$ , and so on. Hence by induction, it will be true for all finite  $n$ .

Let us now proceed to prove the Four Colour Theorem using **Theorem (2)** which has been proved above. We shall prove it using the method of induction.

**Theorem (3):** Four colours are sufficient to colour a map.

Let  $n$  be the number of regions in a map. Assume that four colours will be sufficient to colour the map when  $n = m$ .

As per **Theorem (2)**, in this map with  $m$  regions, three colours will be sufficient to colour the boundary regions. Therefore if we add a new region on the boundary of this map, the added region can be coloured using the fourth colour.

Thus we have seen that the assumption that four colours are sufficient to colour a map for  $n = m$  leads to the conclusion that four colours are sufficient to colour a new map with a newly added region for  $n = (m + 1)$ .

For  $n = 1, 2$  and  $3$ , we do not need four colours to colour the map. We therefore start with  $n = 4$ . For  $n = 4$ , the statement that four colours are sufficient to colour a map is true. Further, three colours are sufficient to colour the boundary regions of this map.

Therefore if we add a new region on the boundary of this map, four colours will be sufficient for  $n = 4 + 1 = 5$ . This means that as the statement is true for  $n = 4$ , it implies that it is true for  $n = 5$ , and so on.

**Hence** by induction, it is proved that four colours are sufficient to colour a map.

### 3. Discussions:

What we have established is a proof of the **Four Colour Theorem** popularly known as **FCT**. The **FCT** was with reference to maps, and we have proved it with reference to maps only. For a map with four regions, four colours are anyway sufficient. We have thereafter shown that to colour the boundary regions of a map with any number of regions, three colours are sufficient. Therefore if on the boundary of a map with four regions a new region is added on the boundary of the map, for the newly added region the unused fourth colour can be used. Thus the **FCT** can be proved by the method of induction.

In figures 1, 2 and 3, we have seen that whereas in the graph  $K_4$  the node A cannot have an edge with a node outside the graph keeping planarity of the graph intact, the maps concerned show that not only the region A (in **Figure -1**) but also the regions A and B (in **Figure-2**) and the regions A, B and C (in **Figure-3**) may be enveloped. As we have seen, a graph hides more than it shows.

In our proof of the **FCT**, we have not used any concept of graph theory. Our proof is not algorithm dependent, and therefore it is not computer dependent. We have narrated logically how the theorem can be proved in such a way that it becomes understandable to everyone. Every time a new region is added on the boundary of a map, we have to colour the boundary regions newly with three colours before proceeding to add a new region on the boundary of the new map.

It is true that how exactly one can colour a given map can actually be an algorithm dependent problem. But this is also true that the theorem was on sufficiency of four colours to colour a map; it was not on how to colour it.

It may be noted that the proof of **Theorem (1)** is a simple proof of the graph theoretic statement that  $K_4$  is a planar graph. There is a standard graph theoretic proof of planarity of  $K_4$  in the text books. Our proof is not based on graph theoretic results.

#### **Acknowledgement:**

We acknowledge that three diagrams from our article cited above [Baruah (2010)] have been included in the present article.



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