



## **AGM, AHM, GHM & AGH: Measures of Central Tendency of Data**

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### **Abstract:**

Derivations of the four measures of average namely Arithmetic-Geometric Mean (abbreviated as AGM), Arithmetic-Harmonic Mean (abbreviated as AHM), Geometric-Harmonic Mean (abbreviated as GHM) and Arithmetic-Geometric-Harmonic (abbreviated as Mean AGHM) as four measures of central tendency of data have been presented in this paper along with numerical examples.

*Key Words:* Numerical Data, Central Tendency, Measure, AGM, AHM, GHM, AGHM

### **1. Introduction:**

Average [Bakker 2003, Chakrabarty 2014 *a,b*] or equivalently mean [Kolmogorov 1930, 1933] is a concept which describes any characteristic of an aggregate / population / class of individuals overall but not of an individual in the aggregate / population / class in particular. It is used in most of the measures associated to data (or list of numerical values). Pythagoras [Christop 2005, David 2004] is the pioneer of defining average. He defined the three most common averages namely arithmetic mean, geometric mean and harmonic mean which were given the name “Pythagorean Means” [Chakrabarty 2016b, 2018b, 2019a,f, 2021b] as a mark of honor to him. Later on, a number of definitions / formulations of average had been derived due to necessity of handling different situations (different types of data) some of which are Quadratic Mean, Square Root Mean, Cubic Mean, Cube Root Mean, Generalized  $p$  Mean & Generalized  $p^{\text{th}}$  Root Mean etc. in addition to Arithmetic Mean, Geometric Mean & Harmonic Mean [Chakrabarty 2016b, 2017a, 2020a].

The next trend was towards developing generalized definitions / formulations of average and general method of defining average. Kolmogorov [Youschkevitch 1983], one great mathematician, formulated



one generalized definition of average known as Generalized  $f$  - Mean [Chakrabarty 2018c , 2018d]. Recently, two generalized definitions of average have been derived among which one is termed as Generalized  $f_H$  – Mean [Chakrabarty 2018e] and the other as Generalized  $f_G$  – Mean [Chakrabarty 2018f, 2019b]. However, none of these three generalized definitions is complete i.e. none of them can describe/yield all types of averages. This leads to the necessity of searching for one general method/definition of average which describe/yield most of the definitions/formulations of average. Accordingly, in another study, an attempt has been made on searching for a generalized method of defining average of a set of values of a variable and later on a generalized method of defining average of a function of a set of values [Chakrabarty 2018h , 2019d , e , 2020a , 2021f].

In statistics, the three Pythagorean means are used in measuring the central tendency of numerical data [Chakrabarty 2015c,d , 2021g , Plackett 1958, Weisberg 1992 , Williams 1984]. However, the accuracy of the value of central tendency yielded by each of the three Pythagorean means is not known. Recently, there have been a lot of studies on analysis of numerical data based on average in general and on Pythagorean means specially [Chakrabarty 2014a,b,c, 2015a, b,c,d,e,g, 2016a , 2017b,c,d, 2018b, 2019g , 2021b,c,f, g,i ]. In the mean time, several attempts have been made on determining accurate value of central tendency of numerical data. However, still there is necessity of more accurate measure of the same. With an objective of finding out more accurate measure of central tendency of data, definitions / formulations of average termed as Arithmetic-Geometric Mean (abbreviated as AGM) [Chakrabarty 2019h , 2020a , 2021a,e, David 2014 , Hazewinkel 2001], Arithmetic-Harmonic Mean (abbreviated as AHM) [Chakrabarty 2020c,d , 2021a,c,d,e], Geometric-Harmonic Mean (abbreviated as GHM) [2020e , 2021a,e] and Arithmetic-Geometric-Harmonic (abbreviated as Mean AGHM) [2020f , 2021a,e,h] respectively have been derived / developed from the three Pythagorean means namely Arithmetic Mean (abbreviated as Mean AM), Geometric Mean (abbreviated as Mean GM) and Harmonic Mean (abbreviated as Mean HM). The derivations of these four formulations of average as four measures of central tendency of data have been presented in this paper along with numerical examples. It is to be mentioned that Arithmetic-Geometric Mean had been defined by Gauss [David 2004 , Hazewinkel 2001] as the point of convergence of two sequences. In a similar manner, Arithmetic-Harmonic Mean [Foster & Phillips 1984] and Geometric-Harmonic Mean were also been defined later on [Chakrabarty 2021e]. However, these three definitions were formulated for two numbers only. In the current study, the definitions of these three along with one more namely Arithmetic-Geometric-Harmonic have been



attempted for a set of finite number of values. The derivations of these four have been presented in this paper with application in numerical data.

**2. AGM, AHM, GHM & AGH:**

Let  $a_0$ ,  $g_0$  &  $h_0$  be respectively the AM, the GM & the HM of the  $N$  numbers (or values or observations)

$$x_1, x_2, \dots, x_N$$

of  $N$  positive numbers or values or observations (not all equal or identical) which are strictly positive

i.e.  $a_0 = AM(x_1, x_2, \dots, x_N) = \frac{1}{N} \sum_{i=1}^N x_i$ , (2.1)

$$g_0 = GM(x_1, x_2, \dots, x_N) = (\prod_{i=1}^N x_i)^{1/N}$$
 (2.2)

$$\& h_0 = HM(x_1, x_2, \dots, x_N) = (\frac{1}{N} \sum_{i=1}^N x_i^{-1})^{-1}$$
 (2.3)

which satisfy the inequality

$$a_0 > g_0 > h_0$$
 (2.4)

[Chakrabarty 2016b, 2018b, 2019a, 201f, 2021b].

The following theorem in the mathematical field of real analysis known as monotone convergence theorem [Weir 1975] will be used in deriving the formulations here.

“If a sequence is increasing and bounded above by a supremum, then the sequence will converge to the supremum; in the same way, if a sequence is decreasing and is bounded below by an infimum, it will converge to the infimum.”

**2. 1. Arithmetic-Geometric Mean (AGM):**

Let the two sequences  $\{a_n\}$  &  $\{g_n\}$  be defined by

$$a_{n+1} = \frac{1}{2}(a_n + g_n)$$
 (2.5)

$$\& g_{n+1} = (a_n \cdot g_n)^{1/2}$$
 (2.6)

respectively where the square root takes the principal value.

It follows from the inequality (2.4) that

$$g_n < a_n$$

Now,  $g_{n+1} = (a_n \cdot g_n)^{1/2} \Rightarrow g_{n+1} > (g_n \cdot g_n)^{1/2} \Rightarrow g_{n+1} > g_n$



This means that the sequence  $\{g_n\}$  is non-decreasing.

Moreover, the sequence  $\{g_n\}$  is bounded above since

$$\text{Smallest of } (x_1, x_2, \dots, x_N) < g_n < \text{Largest of } (x_1, x_2, \dots, x_N)$$

Therefore, by the monotone convergence theorem, the sequence is convergent.

Therefore, there exists a finite number  $M_{AB}$  such that  $g_n$  converges to  $M_{AB}$ .

Again,  $a_n$  can be expressed as

$$a_n = g_{n+1}^2 / g_n$$

This implies that the limiting value of  $a_n$  as  $n$  approaches infinity is  $M_{AB}$ .

Therefore,  $a_n$  also converges to  $M_{AB}$ .

Thus, the two sequences  $\{a_n\}$  &  $\{g_n\}$  converge to a common point  $M_{AB}$ .

**Accordingly**, Arithmetic-Geometric Mean can be defined as follows:

**Definition of Arithmetic-Geometric Mean (AGM):**

The Arithmetic-Geometric Mean (abbreviated by *AGM*) of the  $N$  positive real numbers

$$x_1, x_2, \dots, x_N$$

denoted by  $AGM(x_1, x_2, \dots, x_N)$

is the common limit (or equivalently the common point of convergence)  $M_{AG}$  of the two sequences  $\{a_n\}$  &  $\{g_n\}$  defined by

$$a_{n+1} = \frac{1}{2}(a_n + g_n) \quad \& \quad g_{n+1} = (a_n \cdot g_n)^{1/2}$$

respectively where the square root takes the principal value

and  $a_0$  &  $g_0$  are defined by (2.1) & (2.2) respectively.

**Note:**

(1) It is clear that each of  $\{a_n\}$  &  $\{g_n\}$  converges to  $M_{AG}$ .

Thus, the converging point of either  $\{a_n\}$  or  $\{g_n\}$  can be taken as the value of  $M_{AG}$ .



(2) Since  $g_n < g_{n+1}$ ,

i.e.  $g_0 < g_1 < g_2 < g_3, \dots$

Therefore,  $g_0 < M_{AG}$

Again,  $a_n = g_{n+1}^2 / g_n$  &  $a_{n+1} = g_{n+2}^2 / g_{n+1}$

which implies,  $a_n > a_{n+1}$ ,

i.e.  $a_0 > a_1 > a_2 > a_3, \dots$

Therefore,  $a_0 > M_{AG}$

Hence, the following inequality is obtained:

$$a_0 > M_{AG} > g_0 \quad \text{i.e.} \quad AM > AGM > GM \quad (2.7)$$

## 2.2. Arithmetic-Harmonic Mean (AHM):

Let  $\{a'_n\}$  &  $\{h'_n\}$  be two sequences defined by

$$a'_{n+1} = \frac{1}{2}(a'_n + h'_n) \quad (2.9)$$

$$\& \quad h'_{n+1} = \left\{ \frac{1}{2}(a_n^{-1} + h_n^{-1}) \right\}^{-1} \quad (2.10)$$

respectively.

By inequality (2.1),  $h'_n < a'_n$

Now,  $a'_{n+1} = \frac{1}{2}(a'_n + h'_n) \Rightarrow a'_{n+1} < \frac{1}{2}(a'_n + a'_n) \Rightarrow a'_{n+1} < a'_n$

This means that the sequence  $\{a'_n\}$  is non-increasing.

Moreover, the sequence  $\{a'_n\}$  is bounded below by the smallest of

$$x_1, x_2, \dots, x_N$$

(which follows from the fact that both the arithmetic and the harmonic mean of these numbers lie between the smallest and the largest of them).



Hence, by the monotone convergence theorem, the sequence is convergent.

Therefore, there exists a finite number  $M_{AH}$  such that  $a'_n$  converges to  $M_{AH}$ .

Again,  $h'_n$  can be expressed as

$$h'_n = 2a'_{n+1} - a'_n$$

This implies that the limiting value of  $h'_n$  as  $n$  approaches infinity is  $M_{AH}$ .

Therefore,  $h'_n$  converges to  $M_{AH}$ .

Thus, the two sequences  $\{a'_n\}$  &  $\{h'_n\}$  converge to the same point  $M_{AH}$ .

**Accordingly**, Arithmetic-Harmonic Mean can be defined as follows:

**Definition of Arithmetic-Harmonic Mean (AHM):**

The Arithmetic-Harmonic Mean (abbreviated by AHM) of the  $N$  positive real numbers

$$x_1, x_2, \dots, x_N$$

denoted by  $AHM(x_1, x_2, \dots, x_N)$

is the common limit (or equivalently the common point of convergence)  $M_{AH}$  of two sequences  $\{a'_n\}$  &  $\{h'_n\}$  defined respectively by

$$a'_{n+1} = \frac{1}{2}(a'_n + h'_n) \quad \& \quad h'_{n+1} = \left\{ \frac{1}{2}(a'^{-1}_n + h'^{-1}_n) \right\}^{-1}$$

respectively where  $a_0$  &  $h_0$  are defined by (2.1) & (2.3) respectively.

**Note:**

(1) It is clear that each of  $\{a'_n\}$  &  $\{h'_n\}$  converges to  $M_{AH}$ .

Thus, the converging point of either  $\{a'_n\}$  or  $\{h'_n\}$  can be taken as the value of  $M_{AH}$ .

(2) Since  $a'_{n+1} < a'_n$ ,

i.e.  $a_0 > a'_1 > a'_2 > a'_3, \dots$

Therefore,  $a_0 > M_{AH}$



Again ,  $h'_n = 2\alpha'_{n+1} - \alpha'_n$  &  $h'_{n+1} = 2\alpha'_{n+2} - \alpha'_{n+1}$

which implies,  $h'_n < h'_{n+1}$

i.e.  $h_0 < h'_1 < h'_2 < h'_3 , \dots\dots\dots$

Therefore,  $h_0 \leq M_{AH}$

Hence, the following inequality is obtained:

$$a_0 > M_{AH} > h_0 \quad \text{i.e. } AM > AHM > HM \quad (2.11)$$

### 2. 3. Geometric-Harmonic Mean (GHM):

Let  $\{g''_n\}$  &  $\{h''_n\}$  be two sequences defined respectively by

$$g''_{n+1} = (g''_n \cdot h''_n)^{1/2} \quad (2.13)$$

$$\& \quad h''_{n+1} = \{1/2(g''_n^{-1} + h''_n^{-1})\}^{-1} \quad (2.14)$$

where the square cube takes the principal value.

By inequality (2.1),  $h''_n < g''_n$

and thus  $g''_{n+1} = (h''_n \cdot g''_n)^{1/2} < (g''_n \cdot g''_n)^{1/2} = g''_n$  i.e.  $g''_{n+1} < g''_n$

This means that the sequence  $\{g''_n\}$  is non-increasing.

Moreover, the sequence  $\{g''_n\}$  is bounded below by the smallest of  $x_1, x_2, \dots\dots\dots, x_N$

(which follows from the fact that both the geometric mean and the harmonic mean of these numbers lie between the smallest and the largest of them)..

Hence, by the monotone convergence theorem, the sequence is convergent.

Therefore, there exists a finite number  $M_{GH}$  such that  $g''_n$  converges to  $M_{GH}$ .

Again,

$$h''_n = g''_{n+1}^2 / g''_n$$

This implies that the limiting value of  $h''_n$  as  $n$  approaches infinity is  $M_{GH}$ .



Therefore,  $h''_n$  converges to  $M_{GH}$ .

Thus the two sequences  $\{g''_n\}$  &  $\{h''_n\}$  converge to the same point  $M_{GH}$ .

**This** common limit  $M_{GH}$  can be termed as the Geometric-Harmonic Mean of  $x_1, x_2, \dots, x_N$ .

**Accordingly**, Geometric-Harmonic Mean can be defined as follows:

**Definition of Geometric-Harmonic Mean (GHM):**

The Arithmetic-Harmonic Mean (abbreviated by GHM) of the  $N$  positive real numbers

$$x_1, x_2, \dots, x_N$$

denoted by  $GHM(x_1, x_2, \dots, x_N)$

is the common limit (or equivalently the common point of convergence)  $M_{GH}$  of two the two sequences  $\{g''_n\}$  &  $\{h''_n\}$  defined by

$$g''_{n+1} = (g''_n \cdot h''_n)^{1/2} \quad \& \quad h''_{n+1} = \{1/2(g''_n^{-1} + h''_n^{-1})\}^{-1}$$

respectively where the square root takes the principal value

and  $g_0$  &  $h_0$  are defined by (2.2) & (2.3) respectively.

**Note:**

(1) It is clear that each of  $\{g''_n\}$  &  $\{h''_n\}$  converges to  $M_{GH}$ .

Thus, the converging point of either  $\{g''_n\}$  or  $\{h''_n\}$  can be taken as the value of  $M_{GH}$ .

(2) Since  $g''_{n+1} < g''_n$ ,

i.e.  $g_0 > g''_1 > g''_2 > g''_3, \dots$

Therefore,  $g_0 > M_{GH}$

Again,  $h''_n = g''_{n+1}^2 / g''_n$  which implies,  $h''_n < h''_{n+1}$

i.e.  $h_0 < h''_1 < h''_2 < h''_3, \dots$



Therefore,  $h_0 < M_{GH}$

Hence, the following inequality is obtained:

$$g_0 > M_{GH} > h_0 \quad \text{i.e.} \quad GM > GHM > HM \quad (2.15)$$

#### 2.4. Arithmetic-Geometric-Harmonic Mean (AGHM):

Let us define the three sequences  $\{a''_n\}$ ,  $\{g''_n\}$  &  $\{h''_n\}$  by

$$a''_{n+1} = 1/3 (a''_n + g''_n + h''_n), \quad (2.17)$$

$$g''_{n+1} = (a''_n g''_n h''_n)^{1/3} \quad (2.18)$$

$$\& h''_{n+1} = \{1/3 (a''_n^{-1} + g''_n^{-1} + h''_n^{-1})\}^{-1} \quad (2.19)$$

respectively where the cube root takes the principal value.

By inequality (2.4),

$$h''_n < g''_n < a''_n$$

Now,  $a''_{n+1} < 1/3 (a''_n + a''_n + a''_n)$  i.e.  $a''_{n+1} < a''_n$

This means that the sequence  $\{a''_n\}$  is non-increasing.

Moreover, the sequence  $\{a''_n\}$  is bounded below by the smallest of

$$x_1, x_2, \dots, x_N$$

(which follows from the fact that each of the arithmetic mean, the geometric mean and the harmonic mean of three numbers lies between the smallest and the largest of them).

Hence, by the monotone convergence theorem, the sequence is convergent.

Therefore, there exists a finite number  $c$  such that  $a''_n$  converges to  $c$ .

Again,  $g''_n < a''_n$ , ( $n = 1, 2, 3, \dots$ )

and the sequence  $\{a''_n\}$  is convergent.

Therefore, the sequence  $\{g''_n\}$  is also convergent.

Similarly,  $h''_n < a''_n$ , ( $n = 1, 2, 3, \dots$ ) and the sequence  $\{a''_n\}$  is convergent.

Therefore, the sequence  $\{h''_n\}$  is also convergent.

Now, let us suppose that the two sequences  $\{g''_n\}$  and  $\{h''_n\}$  converge to  $g$  and  $h$  respectively.



Then,  $g'''_n \rightarrow g$  &  $h'''_n \rightarrow h$  as  $n \rightarrow \infty$

Equation (2.17)  $\Rightarrow g'''_n = 3a'''_{n+1} - a'''_n - h'''_n$

& Equation (2.18)  $\Rightarrow h'''_n = g'''_{n+1}{}^{1/3} / (a'''_n g'''_n)$

Therefore,  $g'''_n + g'''_{n+1}{}^3 / (a'''_n g'''_n) = 3 a'''_{n+1} - a'''_n$

Taking limit as  $n \rightarrow \infty$ ,

$$g + \frac{g^3}{c g} = 3c - c \text{ which implies } g^2 + c g - a'''_n = 0$$

This is a quadratic equation in  $g$ .

Solving this equation for  $g$ , it is obtained that

$$g = c \text{ \& } -2c$$

Since  $g$  cannot be negative, therefore we must have  $g = c$

Again from (2.1),  $h'''_n = 3a'''_{n+1} - a'''_n - g'''_n$

Taking limit as  $n \rightarrow \infty$ ,

$$h = 3c - c - g \text{ i.e. } h = 3c - c - c, \text{ since } g = c \text{ which implies } h = c$$

Thus it is obtained that  $g = h = c$

**Therefore**, the three sequences  $\{a'''_n\}$ ,  $\{g'''_n\}$  &  $\{h'''_n\}$  converge to the same limit  $c$ .

**This** common limit can be termed as the Geometric-Harmonic Mean of  $x_1, x_2, \dots, x_N$ .

**Accordingly**, Arithmetic-Geometric-Harmonic Mean can be defined as follows:

**Definition of Arithmetic-Geometric-Harmonic Mean (AGHM):**

The Arithmetic-Geometric-Harmonic Mean (abbreviated by *AGHM*) of the  $N$  positive real numbers

$$x_1, x_2, \dots, x_N$$

denoted by  $AGHM(x_1, x_2, \dots, x_N)$

is the common limit (or equivalently the common point of convergence)  $M_{AGH}$  of the three sequences

$\{a'''_n\}$ ,  $\{g'''_n\}$  &  $\{h'''_n\}$  defined by

$$a'''_{n+1} = 1/3 (a'''_n + g'''_n + h'''_n),$$

$$g'''_{n+1} = (a'''_n g'''_n h'''_n)^{1/3}$$



$$\& h'''_{n+1} = \{1/3 (a'''_n^{-1} + g'''_n^{-1} + h'''_n^{-1})\}^{-1}$$

respectively where the cube root takes the principal value

and  $a_0, g_0$  &  $h_0$  are defined by (2.1), (2.2) & (2.3) respectively.

### 3. Measure of Central Tendency of Data:

If  $\mu$  is the central tendency of the observations

$$x_1, x_2, \dots, x_N$$

then the observations are composed of  $\mu$  and random errors.

In other words, these can be described by or expressed as

$$x_i = \mu + \varepsilon_i, \quad (i = 1, 2, \dots, N)$$

where

$$\varepsilon_1, \varepsilon_2, \dots, \varepsilon_N$$

are the random errors, which assume positive and negative values in random order, associated to

$$x_1, x_2, \dots, x_N$$

respectively.

In this case,

$$AM(x_1, x_2, \dots, x_N) \rightarrow \mu \text{ as } N \rightarrow \infty$$

where 
$$AM(x_1, x_2, \dots, x_N) = \frac{1}{N} \sum_{i=1}^N x_i$$

This implies that  $AM(x_1, x_2, \dots, x_N)$  can be regarded as a measure of the value of  $\mu$  or equivalently the central tendency of the observations  $x_1, x_2, \dots, x_N$ .

Again since the observations

$$x_1, x_2, \dots, x_N$$

consist of  $\mu$  and random errors,

therefore, the reciprocals

$$x_1^{-1}, x_2^{-1}, \dots, x_N^{-1}$$

are composed of  $\mu^{-1}$  and random errors different from the respective random errors



$$\varepsilon_1, \varepsilon_2, \dots, \varepsilon_N$$

provided  $x_1, x_2, \dots, x_N$  are all different from zero.

In this case thus

$$x_i^{-1} = \mu^{-1} + \varepsilon_i' \quad , \quad (i = 1, 2, \dots, N)$$

where

$$\varepsilon_1', \varepsilon_2', \dots, \varepsilon_N'$$

are the random errors, which assume positive and negative values in random order, associated to are the random errors associated to

$$x_1^{-1}, x_2^{-1}, \dots, x_N^{-1}$$

respectively.

In this case,

$$HM(x_1, x_2, \dots, x_N) \rightarrow \mu \text{ as } N \rightarrow \infty$$

where

$$HM(x_1, x_2, \dots, x_N) = \left( \frac{1}{N} \sum_{i=1}^N x_i^{-1} \right)^{-1}$$

This implies that  $HM(x_1, x_2, \dots, x_N)$  can also be regarded as a measure of the value of  $\mu$  or equivalently the central tendency of the observations  $x_1, x_2, \dots, x_N$ .

Again since the observations

$$x_1, x_2, \dots, x_N$$

consist of  $\mu$  and random errors,

these can be described by or expressed as

$$x_i = \mu \varepsilon_i'' \quad , \quad (i = 1, 2, \dots, N)$$

where

$$\varepsilon_1'', \varepsilon_2'', \dots, \varepsilon_N''$$

are the random errors, which assume values in  $(0, 1)$  and in  $(1, \infty)$  in random order, associated to

$$x_1, x_2, \dots, x_N$$

respectively.

In this case,

$$G(x_1, x_2, \dots, x_N) \rightarrow \mu \text{ as } N \rightarrow \infty$$

where

$$G(x_1, x_2, \dots, x_N) = \left( \prod_{i=1}^N x_i \right)^{1/N}$$



This implies that  $GM(x_1, x_2, \dots, x_N)$  can also be regarded as a measure of the value of  $\mu$  or equivalently the central tendency of the observations  $x_1, x_2, \dots, x_N$ .

### 3.1. AGM as a Measure of Central Tendency of Data:

It has already been obtained that the two sequences  $\{a_n\}$  &  $\{g_n\}$  defined by (2.5) & (2.6) respectively converge to a common limit  $M_{AG}$  which is the AGM of  $x_1, x_2, \dots, x_N$ .

Now, from the model it is obtained that

$$a_0 = \mu + \delta_0 \quad \& \quad g_0 = \mu + \xi_0$$

This implies,  $\delta_0 > \xi_0$  since  $a_0 > h_0$

Thus  $a_1 = \mu + \delta_1$  where  $\delta_1 = \frac{1}{2}(\delta_0 + \xi_0) < \delta_0$

In general, corresponding to  $a_{n+1}$ , it holds that

$$\delta_{n+1} = \frac{1}{2}(\delta_n + \xi_n) < \delta_n$$

This implies,  $\delta_n$  becomes smaller and smaller (i.e. closer and closer to 0) as  $n$  larger which implies that  $a_n$  becomes closer to  $\mu$  as  $n$  becomes larger.

Since  $a_n$  and  $g_n$  converge to the same point which is  $AGM(x_1, x_2, \dots, x_N)$

therefore,  $AGM(x_1, x_2, \dots, x_N)$  becomes closer and closer to  $\mu$  as  $n$  becomes larger and larger and hence can be regarded as a measure of the value of  $\mu$  or equivalently the central tendency of the values  $x_1, x_2, \dots, x_N$ .

### 3.2. AHM as a Measure of Central Tendency of Data:

It has already been obtained that the two sequences  $\{a'_n\}$  &  $\{h'_n\}$  defined by (2.9) & (2.10) respectively converge to a common limit  $M_{AH}$  which is the AHM of  $x_1, x_2, \dots, x_N$ .

Now, from the model it is obtained that

$$a_0 = \mu + \delta_0 \quad \& \quad h_0 = \mu + e_0$$

This implies,  $\delta_0 > e_0$  since  $a_0 > h_0$



Thus  $d'_1 = \mu + \delta_1$  where  $\delta_1 = \frac{1}{2}(\delta_0 + e_0) < \delta_0$

In general, corresponding to  $d'_{n+1}$ , it holds that

$$\delta_{n+1} = \frac{1}{2}(\delta_n + e_n) < \delta_n$$

This implies,  $\delta_n$  converges to 0 i.e.  $d'_n$  converges to  $\mu$ .

By the existence of AHM,  $h'_n$  also converges to  $\mu$ .

This implies that the  $AHM(x_1, x_2, \dots, x_N)$  can be regarded as a measure of the value of  $\mu$  or equivalently the central tendency of the observations  $x_1, x_2, \dots, x_N$ .

### 3.3. GHM as a Measure of Central Tendency of Data:

It has already been obtained that the two sequences  $\{g''_n\}$  &  $\{h''_n\}$  defined by (2.13) & (2.14)

respectively converge to a common limit  $M_{GH}$  which is the GHM of  $x_1, x_2, \dots, x_N$ .

Now, from the model it is obtained that

$$g_0 = \mu + \xi_0 \quad \& \quad h_0 = \mu + e_0$$

This implies,  $\xi_0 > e_0$  since  $g_0 > h_0$

Thus  $g''_1 = \mu + \xi''_1$  where  $\xi''_1 = \frac{1}{2}(\xi_0 + e_0) < \xi_0$

In general, corresponding to  $g''_{n+1}$ , it holds that

$$\xi''_{n+1} = \frac{1}{2}(\xi''_n + e''_n) < \xi''_n$$

This implies that  $\xi''_n$  becomes smaller and smaller (i.e. closer and closer to 0) as  $n$  becomes more and more large which implies that  $g''_n$  becomes closer and closer to  $\mu$  as  $n$  becomes larger and larger.

Since  $g''_n$  and  $h''_n$  converge to the same point which is  $GHM(x_1, x_2, \dots, x_N)$

therefore,  $GHM(x_1, x_2, \dots, x_N)$  becomes closer and closer to  $\mu$  as  $n$  becomes larger and larger and hence can be regarded as a measure of the value of  $\mu$  or equivalently the central tendency of the values  $x_1, x_2, \dots, x_N$ .

### 3.4. AGHM as a Measure of Central Tendency of Data:



It has already been obtained that the three sequences  $\{a''''_n\}$ ,  $\{g''''_n\}$  &  $\{h''''_n\}$  defined by (2.17), (2.18) & (2.19) respectively converge to a common limit  $M_{AGH}$  which is the AGHM of  $x_1, x_2, \dots, x_N$ .

Now, from the model it is obtained that

$$a_0 = \mu + \delta_0, \quad g_0 = \mu + \xi_0 \quad \& \quad h_0 = \mu + e_0$$

This implies,  $\delta_0 > \xi_0 > e_0$  since  $a_0 > g_0 > h_0$

Thus  $a''''_1 = \mu + \delta''''_1$  where  $\delta''''_1 = 1/3 (\delta_0 + \xi_0 + e_0) < \delta_0$

In general, corresponding to  $a''''_{n+1}$ , it holds that

$$\delta''''_{n+1} = 1/3 (\delta''''_n + \xi''''_n + e''''_n) < \xi''''_n$$

This implies that  $\delta''''_{n+1}$  becomes smaller and smaller (i.e. closer and closer to 0) as  $n$  becomes more and more large which This implies that  $a''''_n$  becomes closer and closer to  $\mu$  as  $n$  becomes larger more larger.

Since  $a''''_n, g''''_n$  and  $h''''_n$  converge to the same point which is  $AGHM(x_1, x_2, \dots, x_N)$  therefore,  $AGHM(x_1, x_2, \dots, x_N)$  becomes closer and closer to  $\mu$  as  $n$  becomes larger and larger and hence can be regarded as a measure of the value of  $\mu$  or equivalently the central tendency of the values  $x_1, x_2, \dots, x_N$ .

#### 4. Numerical Example – Application to Numerical data:

Observed data considered here are the data on each of annual maximum & annual minimum of surface air temperature, occurred in temperature periodic year (TPR), at Guwahati during the period from 1969 to 2013. The objective here is to evaluate the central tendency of each of annual maximum & annual minimum of surface air temperature at Guwahati

##### 4.1. Annual Maximum of Surface Air Temperature at Guwahati:

From the observed data on annual maximum of surface air temperature, occurred in temperature periodic year (TPR), at Guwahati during the period from 1969 to 2013, the values (in Degree Celsius) of  $AM, GM$  &  $HM$  have been found as follows:



$$AM = a_0 = 37.2093023255814 ,$$

$$GM = g_0 = 37.1922871485760$$

$$\& HM = h_0 = 37.17539890356262$$

Here the observed values can be assumed to be composed of a parameter  $\mu$  (representing the central tendency of annual maximum) and random errors.

**Here** the observed values can be assumed to be composed of a parameter  $\mu$  (representing the central tendency of annual maximum) and random errors.

**Evaluation of Value of  $\mu$  (the central tendency of annual maximum) by AGM:**

In this case the iterations give the values as shown in the following table (Table – 4.1.1):

**Table – 4.1.1**

| $n$ | Value of $a_n$                           | Value of $g_n$                           |
|-----|------------------------------------------|------------------------------------------|
| 1   | <u>37.200794737078738390962906373793</u> | <u>37.20079376426265222211622224967</u>  |
| 2   | <u>37.200794250670695306539564311732</u> | <u>37.200794250670692126596916004529</u> |
| 3   | <u>37.200794250670693716568240158131</u> | <u>37.20079425067069371656824015813</u>  |

AGM = The common point of converging = 37.20079425067069371656824015813

**Evaluation of Value of  $\mu$  (the central tendency of annual maximum) by AHM:**

In this case the iterations give the values as shown in the following table (Table – 4.1.2):

**Table – 4.1.2**

| $n$ | Value of $a_n$                           | Value of $h_n$                           |
|-----|------------------------------------------|------------------------------------------|
| 1   | <u>37.192350614572013817418147245751</u> | <u>37.183872827043641055199463981829</u> |
| 2   | <u>37.18811172080782743630880561379</u>  | <u>37.188111237636740962336357297681</u> |
| 3   | <u>37.188111479222284199322581455736</u> | <u>37.188111479222282629907723330485</u> |
| 4   | <u>37.188111479222283414615152393111</u> | <u>37.188111479222283022261437861789</u> |
| 5   | <u>37.18811147922228321843829512745</u>  | <u>37.188111479222283218438295127448</u> |
| 6   | <u>37.188111479222283218438295127449</u> | <u>37.188111479222283218438295127449</u> |



$AHM =$  The common point of converging = 37.188111479222283218438295127449 Accordingly, the value of  $\mu$ , the central tendency of annual maximum of surface air temperature at Guwahati, obtained by  $AHM$ , is 37.188111479222283218438295127449 Degree Celsius.

**Evaluation of Value of  $\mu$  (the central tendency of annual maximum) by GHM:**

In this case the iterations give the values as shown in the following table (Table – 4.1.3):

**Table – 4.1.3**

| $n$ | Value of $a''_n$                         | Value of $h''_n$                         |
|-----|------------------------------------------|------------------------------------------|
| 1   | <u>37.183842067276499922160771566921</u> | <u>37.183841108483672358606539675987</u> |
| 2   | <u>37.183841587880083050050439193677</u> | <u>37.183841587880079959717222765898</u> |
| 3   | <u>37.183841587880081504883830979787</u> | <u>37.183841587880081504883830979784</u> |
| 4   | <u>37.183841587880081504883830979786</u> | <u>37.183841587880081504883830979786</u> |

$GHM =$  The common point of converging = 37.183841587880081504883830979786

Thus the central tendency of annual maximum of surface air temperature at Guwahati, obtained by  $GHM$ , is 37.183841587880081504883830979786 Degree Celsius.

**Evaluation of Value of  $\mu$  (the central tendency of annual maximum) by AGHM:**

In this case the iterations give the values as shown in the following table (Table – 4.1.4):

**Table – 4.1.4**

| $n$ | Term of Sequence $a'''_n, g'''_n$ & $h'''_n$ | Value                                    |
|-----|----------------------------------------------|------------------------------------------|
| 1   | $a'''_1$                                     | <u>37.192329459240034805587369079696</u> |
|     | $g'''_1$                                     | <u>37.192326883784773277226087433254</u> |
|     | $h'''_1$                                     | <u>37.192324308332441617854668614447</u> |
| 2   | $a'''_2$                                     | <u>37.192326883785749900222708375799</u> |
|     | $g'''_2$                                     | <u>37.192326883785690452815011296956</u> |
|     | $h'''_2$                                     | <u>37.192326883785631005407314219677</u> |
| 3   | $a'''_3$                                     | <u>37.192326883785690452815011297477</u> |
|     | $g'''_3$                                     | <u>37.192326883785690452815011297446</u> |
|     | $h'''_3$                                     | <u>37.192326883785690452815011297413</u> |
| 4   | $a'''_4$                                     | <u>37.192326883785690452815011297445</u> |
|     | $g'''_4$                                     | <u>37.192326883785690452815011297445</u> |
|     | $h'''_4$                                     | <u>37.192326883785690452815011297441</u> |



|   |              |                                   |
|---|--------------|-----------------------------------|
| 5 | $\alpha''_5$ | 37.192326883785690452815011297444 |
|   | $g'''_5$     | 37.192326883785690452815011297444 |
|   | $h'''_5$     | 37.192326883785690452815011297444 |

$AGHM = \text{The common point of converging} = 37.192326883785690452815011297444$

Thus the value of  $\mu$ , the central tendency of annual maximum of surface air temperature at Guwahati, obtained by  $AGHM$ , is 37.192326883785690452815011297444 Degree Celsius.

#### 4.2. Annual Minimum of Surface Air Temperature at Guwahati:

From the observed data on annual maximum of surface air temperature, occurred in temperature periodic year (TPR), at Guwahati during the period from 1969 to 2013, the values (in Degree Celsius) of  $AM$ ,  $GM$  &  $HM$  have been found as follows:

$$AM = \alpha_0 = 7.3634146341463414634146341463415 ,$$

$$GM = g_0 = 7.2597176194576185608709616351297$$

$$\& HM = h_0 = 7.1543933802823525209849744707569$$

In this case also, the observed values can be assumed to be composed of a parameter  $\mu$  (representing the central tendency of annual maximum) and random errors.

#### Evaluation of Value of $\mu$ (the central tendency of annual maximum) by AGM:

In this case the iterations give the values as shown in the following table (Table – 4.2.1):

**Table – 4.2.1**

| $n$ | Value of $\alpha_n$                      | Value of $g_n$                           |
|-----|------------------------------------------|------------------------------------------|
| 1   | <u>7.3115661268019800121427978907356</u> | <u>7.311382287836156454734736062194</u>  |
| 2   | <u>7.3114742073190682334387669764648</u> | <u>7.3114742067412646948141851474363</u> |
| 3   | <u>7.3114742070301664641264760619506</u> | <u>7.3114742070301664641207683052145</u> |
| 4   | <u>7.3114742070301664641236221835826</u> | <u>7.3114742070301664641236221835825</u> |
| 5   | <u>7.3114742070301664641236221835825</u> | <u>7.3114742070301664641236221835825</u> |
| 6   | <u>7.3114742070301664641236221835825</u> | <u>7.3114742070301664641236221835825</u> |

$AGM = \text{The common point of convergence} = 7.3114742070301664641236221835825$



Thus the value of  $\mu$ , the central tendency of annual minimum of surface air temperature at Guwahati, obtained by *AGM*, is 7.3114742070301664641236221835825 Degree Celsius.

**Evaluation of Value of  $\mu$  (the central tendency of annual minimum) by *AHM*:**

In this case the iterations give the values as shown in the following table (Table – 4.2.2):

**Table – 4.2.2**

| $n$ | Value of $a'_n$                          | Value of $h'_n$                          |
|-----|------------------------------------------|------------------------------------------|
| 1   | <u>7.2589040072143469921998043085492</u> | <u>7.2573993074510131470508335954916</u> |
| 2   | <u>7.2581516573326800696253189520204</u> | <u>7.2581515793472133602889381001075</u> |
| 3   | <u>7.258151618339946714957128526064</u>  | <u>7.2581516183399465054777273757201</u> |
| 4   | <u>7.2581516183399466102174279508921</u> | <u>7.2581516183399466102174279508919</u> |
| 5   | <u>7.258151618339946610217427950892</u>  | <u>7.258151618339946610217427950892</u>  |

*AHM* = The common point of convergence = 7.258151618339946610217427950892

Thus the central tendency of annual minimum of surface air temperature at Guwahati, obtained by *AHM*, is 7.258151618339946610217427950892 Degree Celsius.

**Evaluation of Value of  $\mu$  (the central tendency of annual maximum) by *GHM*:**

In this case the iterations give the values as shown in the following table (Table – 4.2.3):

**Table – 4.2.3**

| $n$ | Value of $g''_n$                         | Value of $h''_n$                         |
|-----|------------------------------------------|------------------------------------------|
| 1   | <u>7.2068630956447857997320179691161</u> | <u>7.2066706965561339698683103736099</u> |
| 2   | <u>7.2067668954583999665077195028225</u> | <u>7.2067668948163400482724767591701</u> |
| 3   | <u>7.2067668951373700073829478903856</u> | <u>7.2067668951373700073757976497748</u> |
| 4   | <u>7.2067668951373700073793727700802</u> | <u>7.2067668951373700073793727700802</u> |

*GHM* = The common point of convergence = 7.2067668951373700073793727700802

Thus the central tendency of annual minimum of surface air temperature at Guwahati, obtained by *GHM*, is 7.2067668951373700073793727700802 Degree Celsius.



**Evaluation of Value of  $\mu$  (the central tendency of annual minimum) by AGHM:**

In this case the iterations give the values as shown in the following table (Table – 4.2.4):

From this table it is found that

$$AGHM = \text{The common point of convergence} = 7.2586735571288657555393158774538$$

Thus the central tendency of annual minimum of surface air temperature at Guwahati, obtained by AGHM, is 7.2586735571288657555393158774538 Degree Celsius.

**Table – 4.2.4**

| $n$ | Term of Sequence $a'''_n, g'''_n$ & $h'''_n$ | Value                                    |
|-----|----------------------------------------------|------------------------------------------|
| 1   | $a'''_1$                                     | <u>7.259175211295437515090190084076</u>  |
|     | $g'''_1$                                     | <u>7.2586735811751601863075880738685</u> |
|     | $h'''_1$                                     | <u>7.2581719135850851422025166245462</u> |
| 2   | $a'''_2$                                     | <u>7.2586735686852276145334315941636</u> |
|     | $g'''_2$                                     | <u>7.2586735571288657683004939856174</u> |
|     | $h'''_2$                                     | <u>7.258673545572503902182631426563</u>  |
| 3   | $a'''_3$                                     | <u>7.2586735571288657616721856687813</u> |
|     | $g'''_3$                                     | <u>7.2586735571288657555393158774539</u> |
|     | $h'''_3$                                     | <u>7.258673557128865749406446086127</u>  |
| 4   | $a'''_4$                                     | <u>7.2586735571288657555393158774541</u> |
|     | $g'''_4$                                     | <u>7.2586735571288657555393158774539</u> |
|     | $h'''_4$                                     | <u>7.2586735571288657555393158774538</u> |
| 5   | $a'''_5$                                     | <u>7.2586735571288657555393158774539</u> |
|     | $g'''_5$                                     | <u>7.2586735571288657555393158774538</u> |
|     | $h'''_5$                                     | <u>7.2586735571288657555393158774538</u> |
| 6   | $a'''_6$                                     | <u>7.2586735571288657555393158774538</u> |
|     | $g'''_6$                                     | <u>7.2586735571288657555393158774538</u> |
|     | $h'''_6$                                     | <u>7.2586735571288657555393158774538</u> |

**5. Conclusion:**

From the description, presented above, one can arrive at the following conclusions:

- (1) In addition to AM, GM, & HM, each of the four formulations namely AGM, AHM, GHM & AGHM can be regarded as a measure of average of a set of numbers.



(2) AGM, AHM, GHM & AGHM are defined only when the associated numbers are strictly positive. For numbers other than strictly positive, there is necessity of searching for technique of finding these types of average.

(3) Each of AGM, AHM, GHM & AGHM can be accepted as measure of central tendency of data in addition to its usual measures namely AM, GM, & HM.

(4) Each of AM, GM, & HM becomes closer and closer to the value of the central tendency of data if the data size becomes larger and larger. On the other hand, even if for data of small size, each of AGM, AHM, GHM & AGHM is very close to the value of the central tendency of data, Thus, these four can be treated as improved measures of central tendency of data.

Finally, from the meaning of research [Chakrabarty 2018a,g , 2019c], one can conclude that the discovery of *AGM*, *AHM*, *GHM* & *AGHM* as measures of central tendency of data can be regarded as research findings carrying fundamental importance and high significance in the theory of analysis of data specially of measure of central tendency of data.



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