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Arithmetic-Harmonic Mean: Evaluation of Parameter from Observed Data Containing Itself and Random Error

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Abstract

Some studies have already been made on method of determining the value of parameter from observed data containing the parameter itself and random error due to the reason that the existing statistical methods of estimation in such situation fail in finding out the appropriate value of the parameter. The methods, so developed, suffer from two limitations which are: (i) the methods involve huge computational tasks and (ii) a finite set of observed data may not yield the appropriate value of the parameter in many situations while the number of observations required in the methods may be too large for obtaining the appropriate value of the parameter. For these two limitations one method, based on arithmetic-geometric mean, for the same has recently been developed which involves lesser computational tasks than those involved in the methods developed earlier and which can be applicable in the case of finite set of data. In this paper, another method has been developed for the same which is based on arithmetic-harmonic mean. This paper describes the derivation of the method and one numerical application of the method in determining the central tendency of each of annual maximum and annual minimum of surface air temperature at Guwahati.

Keywords: AHM , parameter, random error, evaluation of parameter, surface air temperature,;central tendency.

1. Introduction

In the situation where the numerical data

$$x_1\ ,\, x_2\ ,\ \ldots\ldots\ldots\ldots\ ,\, x_N$$

are composed of some parameter μ and random errors ε_i which means

$$x_i = \mu + \varepsilon_i \;\;, \quad (i = 1, 2, \ldots\ldots\ldots\ldots, N) \qquad \text{---------------------} \quad (1.1)$$

[Chakrabarty (2014*a* , 2014*b* , 2014*c* , 2015*a* , 2015*b* , 2015*c* , 2015*d* , 2015*e* , 2015*f* , 2015*g* , 2017*b*)], the existing methods of estimation namely least squares method, maximum likelihood method, minimum variance unbiased method, method of moment and method of minimum chi-square, [Aldrich (2000.**),** Anders (1999) , Barnard (1949), Birnbaum (1962), Ivory (1825) , Kendall & Stuart (1977), Lehmann & Casella George (1998), Lucien (1990) , Walker & Lev (1965)] provides $\frac{1}{N} \sum_{i=1}^{N} x_i$ as estimator of the parameter μ which suffers from an error $\frac{1}{N} \sum_{i=1}^{N} \varepsilon_i$ [Chakrabarty (2014*a* , 2014*b* , 2014*c*)].

In some recent studies, some methods have been developed for determining the appropriate value of the parameter μ involved in the model described by equation(1.1) [Chakrabarty (2014*a* , 2014*b* , 2014*c* , 2015*a* , 2015*b* , 2015*c* , 2015*d* , 2015*e* , 2015*f* , 2015*g* , 2016*a* , 2016*b* , 2019*a* , 2019*c*]. In these studies some methods have been developed for determining the appropriate value of the parameter μ when ε_i occurs due to random cause.

The first method, developed for the same is based on computing sequence of interval value of μ with decreasing length of interval and then to find out the shortest interval value of μ [Chakrabarty (2014*a* , 2014*b* , 2014*c* , 2015*d*)] while the second one is based on stable mid range and median (Chakrabarty , 2015*b*) and the third one on the convergence of statistic i.e. some function of the available numerical data (Chakrabarty , 2017*a*). The fourth one (Chakrabarty , 2018*a*) has been developed on the basis of Pythagorean means [Kolmogorov (1930) , O'Meara (1989) , Riedweg (2005) , Cornelli, McKirahan & Macris (2013) , de Carvalho (2016) , Chakrabarty (2018*b* , 2018*c* , 2018*d* , 2018*e* , 2018*f* , 2019*d* , 2019*e* , 2019*f*], 2020*a*)] while the fifth one [Chakrabarty (2016*c* , 2019*b*)] for the same is based on the probabilistic convergence of Pythagorean means [Chakrabarty (2017*b* , 2017*b*)}.

The methods, developed so far, for determining the appropriate value of the parameter from observed data containing the parameter itself and random error involve huge computational tasks. Moreover, a finite set of observed data may not yield the appropriate value of the parameter in many situations while the number of observations required in the methods may be too large for obtaining the appropriate value of the parameter. For these two probable drawbacks, one method for the same purpose was developed which involves lesser computational tasks than those involved in the earlier methods as well as which can be applicable in the case of finite set of data [Chakrabarty (2019*g* , 2020*b*)]. The method developed is based on the concept of Arithmetic-Geometric Mean. In the current attempt, another method has been attempted for the same purpose which is based on the concept of Arithmetic-Geometric Mean.This paper describes the derivation of the method and one numerical application of the method in determining the central tendency of each of annual maximum and annual minimum of surface air temperature at Guwahati.

2. Arithmetic-Harmonic Mean (AHM)

Let a_0 & h_0 be respectively the *AM* (Arithmetic Mean) & *HM* (Harmonic Mean) of the *N* numbers (or values or observations)

$$x_1 , x_2 , \ldots\ldots\ldots\ldots , x_N$$

From the inequality of Pythagorean means [Chakrabarty (2016*c*), Cornelli (2013)] namely

$$AM > GM > HM$$

(where GM means Geometric Mean), it follows that

$$a_0 > h_0$$

provided $x_1 , x_2 , \ldots\ldots\ldots\ldots , x_N$ are not all equal.

Let $\{a'_n = a'_n(a_0 , h_0)\}$ & $\{h'_n = h'_n(a_0 , h_0)\}$ be two sequences defined by

$$a'_{n+1} = ½(a'_n + h'_n) \ \&\ h'_{n+1} = ½(a'^{-1}_n + h'^{-1}_n)\}^{-1} \text{ respectively.}$$

It is obvious that

$$a'_0 = a'_0(a_0 , h_0) = a_0 \quad \& \quad h'_0 = h'_0(a_0 , h_0) = h_0$$

By the inequality of Pythagorean means [Chakrabarty (2016*c*), Cornelli (2013)],

$$h'_n < a'_n$$

and thus $\qquad a'_{n+1} = ½ (a'_n + h'_n)$

$$\Rightarrow a'_{n+1} < ½(a'_n + a'_n)$$

$$\Rightarrow a'_{n+1} < a'_n$$

This means that the sequence $\{a'_n = a'_n(a_0 , h_0)\}$ is non-increasing.

Moreover, the sequence $\{a'_n = a'_n(a_0 , h_0)\}$ is bounded below by the smallest of

$$x_1\,,x_2\,,\ \ldots\ldots\ldots\ldots\,,x_N$$

(which follows from the fact that both the arithmetic and the harmonic mean of these numbers lie between the smallest and the largest of them).

In the mathematical field of real analysis, the monotone convergence theorem [Weir (1973) , Yeh (2006)] states that if a sequence is increasing and bounded above by a supremum, then the sequence will converge to the supremum; in the same way, if a sequence is decreasing and is bounded below by an infimum, it will converge to the infimum.

Thus, by the monotone convergence theorem, the sequence is convergent. Therefore, there exists a finite number M_{AH} such that,

$$a'_n \text{ converges to } M_{AH} \text{ as } n \text{ approaches infinity.}$$

Again, h'_n can be expressed as

$$h'_n = 2a'_{n+1} - a'_n$$

This implies that the limiting value of h'_n as n approaches infinity is M_{AH} .

Therefore,

$$h'_n \text{ converges to } M_{AH} \text{ as } n \text{ approaches infinity.}$$

Thus, the two sequences $\{a'_n = a'_n(a_0\,,\ h_0)\}$ & $\{h'_n = h'_n(a_0\,,\ h_0)\}$ converge to the same point M_{AH} as n approaches infinity.

This common converging point M_{AH} can be termed / named / regarded as the Arithmetic-Harmonic Mean (abbreviated as *AHM*) of the N numbers (or values or observations)

$$x_1\,,x_2\,,\ \ldots\ldots\ldots\ldots\,,x_N$$

Accordingly, *AHM* can be defined as follows:

2.1. Definition of Arithmetic-Harmonic Mean (AHM)

If a_0 & h_0 are respectively the AM & the HM of N numbers (or values or observations) viz $x_1\,,x_2\,,\ \ldots\ldots\ldots\ldots\,,x_N$

Then the two sequences $\{a'_n = a'_n(a_0\,,\ h_0)\}$ & $\{h'_n = h'_n(a_0\,,\ h_0)\}$ defined respectively by

$$a'_{n+1} = ½(a'_n + h'_n) \ \&\ h'_{n+1} = ½(a'^{-1}_n + h'^{-1}_n)\}^{-1}$$

where $a'_0 = a'_0(a_0, h_0) = a_0$ & $h'_0 = h'_0(a_0, h_0) = h_0$ converge to a common limit M_{AH} which is called the Arithmetic-Harmonic Mean (abbreviated by *AHM*) of $x_1, x_2, \ldots\ldots\ldots, x_N$ and is denoted here by $AHM(x_1, x_2, \ldots\ldots\ldots, x_N)$ i.e.

$AHM(x_1, x_2, \ldots\ldots\ldots, x_N) = M_{AH}$

Note:

(1) It is clear that each of $\{a'_n\}$ & $\{h'_n\}$ converges to M_{AH} .

Thus, the converging point of either $\{a'_n\}$ or $\{h'_n\}$ can be taken as the value of M_{AH}.

(2) Since $a'_{n+1} \leq a'_n$,

i.e. $a_0 \geq a'_1 \geq a'_2 \geq a'_3$, ...

Therefore, $a_0 \geq M_{AH}$

Again , $h'_n = 2a'_{n+1} - a'_n$

& $h'_{n+1} = 2a'_{n+2} - a'_{n+1}$

which implies, $h'_n \leq h'_{n+1}$

i.e. $h_0 \leq h'_1 \leq h'_2 \leq h'_3$, ...

Therefore, $h_0 \leq M_{AH}$

Hence, the following inequality is obtained:

$a_0 \geq M_{AH} \geq h_0$

This can be stated as follows:.

$AM \geq AHM \geq HM.$

3. Evaluation of μ

If the observations

$x_1 , x_2 , \ldots\ldots\ldots , x_N$

are composed of some parameter μ and random errors then the observations can be expressed as

$$x_i = \mu + \varepsilon_i \quad , \quad (i = 1 , 2 , \ldots\ldots\ldots , N)$$

where

$\varepsilon_1 , \varepsilon_2 , \ldots\ldots\ldots , \varepsilon_N$

are the random errors, which assume positive and negative values in random order, associated to

$$x_1 , x_2 , \ldots\ldots\ldots , x_N$$

respectively.

In this case,

$$A(x_1 , x_2 , \ldots\ldots\ldots , x_N) \rightarrow \mu \text{ as } N \rightarrow \infty$$

where $$A(x_1 , x_2 , \ldots\ldots\ldots , x_N) = \frac{1}{N} \sum_{i=1}^{N} x_i$$

Again since the observations

$x_1 , x_2 , \ldots\ldots\ldots , x_N$

consist of μ and random errors,

therefore, the reciprocals

$x_1^{-1} , x_2^{-1} , \ldots\ldots\ldots\ldots , x_N^{-1}$

are composed of μ^{-1} and random errors different from the respective random errors

$\varepsilon_1 , \varepsilon_2 , \ldots\ldots\ldots , \varepsilon_N$

provided $x_1 , x_2 , \ldots\ldots\ldots , x_N$ are all different from zero.

In this case thus

$$x_i^{-1} = \mu^{-1} + \varepsilon_i' \quad , \quad (i = 1 , 2 , \ldots\ldots\ldots , N)$$

where

$\varepsilon_1' , \varepsilon_2' , \ldots\ldots\ldots\ldots , \varepsilon_N'$

are the random errors, which assume positive and negative values in random order, associated to are the random errors associated to

x_1^{-1}, x_2^{-1} , …………… , x_N^{-1}

respectively..

In this case,

$$H(x_1 , x_2 , \ldots\ldots\ldots , x_N) \longrightarrow \mu \text{ as } N \longrightarrow \infty$$

where $$H(x_1 , x_2 , \ldots\ldots\ldots , x_N) = \left(\frac{1}{N} \sum_{i=1}^{N} x_i^{-1}\right)^{-1}$$

This implies that the common converging value of $A(x_1 , x_2 , \ldots\ldots\ldots , x_N)$ and $H(x_1 , x_2 , \ldots\ldots\ldots , x_N)$ is the value of μ.

It is to be noted that the converging value may not be possible to be obtained for a finite set of observed values namely

$x_1 , x_2 , \ldots\ldots\ldots , x_N$

In order to obtain the value of μ, in this case, let us write

$$A(x_1 , x_2 , \ldots\ldots\ldots , x_N) = A_0$$

$$\& \; H(x_1 , x_2 , \ldots\ldots\ldots , x_N) = H_0$$

and then define the two interdependent sequences $\{A_n\}$ and $\{H_n\}$ asrespectively by

$A_{n+1} = ½ (A_n + H_n)$

& $H_{n+1} = \{½ (A_n^{-1} + H_n^{-1})\}^{-1}$

Then, both of A_n & H_n converges to some real number C as *n* approaches infinity.

Now, it is required to verify whether this C is equal to μ.

From the model it is obtained that

$A_0 = \mu + \delta_0$ & $H_0 = \mu + e_0$

The inequality of Pythagorean means namely

$$AM > HM$$

implies that $A_0 > H_0$ i.e. $\delta_0 > e_0$

Thus $A_1 = \mu + \delta_1$ where $\delta_1 = ½\,(\delta_0 + e_0) < \delta_0$

In general, corresponding to A_{n+1}, it holds that

$$\delta_{n+1} = ½\,(\delta_n + e_n) < \delta_n$$

This implies, δ_n converges to 0 i.e. A_n converges to μ.

By the existence of AHM, H_n also converges to μ.

Thus, the AHM of $x_1, x_2, \ldots\ldots, x_N$ is the value of μ.

4. Application to Numerical Data

Observed data considered here are the data on each of annual maximum & annual minimum of surface air temperature, occurred in temperature periodic year (TPR), at Guwahati during the period from 1969 to 2013. The objective here is to evaluatethe central tendency of each of annual maximum & annual minimum of surface air temperature at Guwahati

4.1. Annual Maximum of Surface Air Temperature at Guwahati:

The following table (**Table – 4.1.1**) shows the observed data on annual maximum of surface air temperature, occurred in temperature periodic year (TPR), at Guwahati during the period from 1969 to 2013:

Table – 4.1.1

Observed Value on Annual Maximum of Surface Air Temperature (in Degree Celsius)

TPR No (i)	Observed Value (x_i)	TPR No (i)	Observed Value (x_i)	TPR No (i)	Observed Value (x_i)	TPR No (i)	Observed Value (x_i)
1	37.1	12	35.1	23	37.4	34	38.0
2	36.6	13	35.8	24	39.4	35	36.6
3	36.0	14	36.5	25	36.4	36	38.0
4	35.7	15	36.7	26	38.1	37	37.3

5	39.0	16	37.2	27	36.3	38	37.3
6	36.1	17	36.5	28	39.9	39	38.0
7	39.2	18	38.4	29	37.4	40	37.2
8	39.0	19	37.2	30	37.5	41	37.3
9	35.3	20	36.4	31	36.7	42	37.4
10	36.8	21	36.7	32	35.7	43	38.8
11	38.6	22	36.0	33	37.4		

Here the observed values x_i (i = 1 , 2 , 3 , ... , 43) can be assumed to be composed of a parameter μ (representing the central tendency of annual maximum) and random errors so that the observed values follow the model described by equation (1.1).

Evaluation of Value of μ (the central tendency of annual maximum)

The computed values of the arithmetic mean and the geometric mean of the observed values, shown in **Table – 4.1.1**, are found to be 37.2093023255814 and 37.175398903562627634836294491501 respectively.

Let us write

A_0 = 37.2093023255814 & H_0 = 37.175398903562627634836294491501

In this case the iterations give the values which are given in the following table (**Table – 4.1.2**):

Table – 4.1.2

n	A_n	H_n
0	37.2093023255814	37.175398903562627634836294491501
1	37.192350614572013817418147245751	37.183872827043641055199463981829
2	37.18811172080782743630880561379	37.188111237636740962336357297681
3	37.188111479222284199322581455736	37.188111479222282629907723330485
4	37.188111479222283414615152393111	37.188111479222283022261437861789
5	37.18811147922228321843829512745	37.188111479222283218438295127448
6	37.188111479222283218438295127449	37.188111479222283218438295127449

The digits in A_n and H_n, which are agreed, have been underlined in the above table.

The AHM of 37.2093023255814 and 37.1753989035626276348362944915O1is the common limit of these two sequences which is 37.1881114792222832184382951274449.

Thus the value of μ, the central tendency of annual maximum of surface air temperature at Guwahati, obtained by AHM, is 37.1881114792222832184382951274449 Degree Celsius.

4.2. Annual Minimum of Surface Air Temperature at Guwahati:

The following table (**Table – 4.2.1**) shows the observed data on annual maximum of surface air temperature, occurred in temperature periodic year (TPR), at Guwahati during the period from1969 to 2013.

As earlier, the observed values

x_i (i = 1 , 2 , 3 , ... , 44)

can in this case also be assumed to be composed of a parameter μ (representing the central tendency of annual minimum) and random errors so that the observed values follow the model described by equation (1.1).

Table – 4.2.1

Observed Value on Annual Minimum of Surface Air Temperature (in Degree Celsius)

TPR No (i)	Observed Value (x_i)	TPR No (i)	Observed Value (x_i)	TPR No (i)	Observed Value (x_i)	TPR No (i)	Observed Value (x_i)
1	6.6	12	6.4	23	7.4	34	8.0
2	6.6	13	7.5	24	5.9	35	7.9
3	5.9	14	8.3	25	8.4	36	6.7
4	8.2	15	4.9	26	7.8	37	9.6
5	5.0	16	6.1	27	7.5	38	6.4
6	6.3	17	7.8	28	9.4	39	7.8
7	7.4	18	8.6	29	NA	40	9.9

8	6.6	19	7.7	30	NA	41	8.6
9	6.2	20	9.2	31	NA	42	7.0
10	7.3	21	6.7	32	8.9	43	6.4
11	6.2	22	8.6	33	8.6	44	5.6

Determination of Value of μ (the central tendency of annual minimum)

The computed values of arithmetic mean and the harmonic mean of the observed values, shown in **Table –4.2.1**, are found to be 7.363414634146341463414634146341463415 and 7.1543933802823525209849744707569 respectively.

Let us write

A_0 = 7.363414634146341463414634146341463415 & H_0 = 7.1543933802823525209849744707569

In this case the iterations give the values which are given in the following table (**Table – 4.2.2**):

Table – 4.2.2

n	A_n	H_n
0	7.363414634146341463414634146341463415	7.1543933802823525209849744707569
1	7.2589040072143469921998043085492	7.2573993074510131470508335954916
2	7.2581516573326800696253189520204	7.2581515793472133602889381001075
3	7.258151618339946714957128526064	7.2581516183399465054777273757201
4	7.2581516183399466102174279508921	7.2581516183399466102174279508919
5	7.258151618339946610217427950892	7.258151618339946610217427950892

The digits in A_n and H_n, which are agreed, have been underlined in the above table.

The AHM of 7.363414634146341463414634146341463415 and 7.1543933802823525209849744707569 is the common limit of these two sequences which is 7.258151618339946610217427950892.

Thus the value of μ, the central tendency of annual minimum of surface air temperature at Guwahati, obtained by AHM, is 7.2581516183399466102174279508922 Degree Celsius.

5. Summary of computations

The methods developed so far, for determining thr appropriate value of the parameter from observed data containing the parameter itself and random error involve huge computational tasks. The method, based on AHM, described here involves lesser computational tasks than those involved in the methods developed so far.

Moreover, a finite set of observed data may not yield the appropriate value of the parameter in many situations while the number of observations required in the methods may be too large for obtaining the appropriate value of the parameter. The method, based on AHM, can be applicable in the case of finite set of data.

Regarding the findings obtained on annual maximum and annual minimum of surface air temperature at Guwahati, the following conclusion can be drawn:

5.1. The value of central tendency of annual maximum of surface air temperature at Guwahati, obtained by AHM, is 37.188111479222283218438295127449 Degree Celsius. However, the value of the same was found to be 37.2 Degree Celsius by the earlier methods of the same [2014*c* , 2015*a* , 2015*b* , 2015*c* , 2015*d* , 2015*e* , 2017 , 2018*a* 2019*b*] while the value ofr the same obtained by AGM is 37.20079425067069371656824015813 Degree Celsius (Chakrabarty, 2020*b*).

5.2. The central tendency of annual maximum of surface air temperature at Guwahati, obtained by AHM, is 7.2581516183399466102174279508922 Degree Celsius. However, it was found not possible to obtain the value of central tendency of the same by the earlier methods [2014*c* , 2015*a* , 2015*b* , 2015*c* , 2015*d* , 2015*e* , 2017 , 2018*a* 2019*b*] while the value for the same obtained by AGM is 7.0100883193127795038175836675175 Degree Celsius (Chakrabarty, 2020*b*).

Reference

Aldrich John, Fisher's Inverse Probability of 1930, *International Statistical Review*, Vol. 68, pp. 155 – 172, 2000.

Anders Hald, On the History of Maximum Likelihood in Relation to Inverse Probability and Least Squares, *Statistical Science*, Vol. 14, pp. 214 – 222, 1999.

Barnard G. A., Statistical Inference, *Journal of the Royal Statistical Society, Series B*, Vol. 11, pp. 115 – 149, 1949.

Birnbaum Allan, On the Foundations of Statistical Inference, *Journal of the American Statistical Association*, Vol. 57, pp. 269 – 306, 1962.

Chakrabarty Dhritikesh, Determination of Parameter from Observations Composed of Itself and Errors, *International Journal of Engineering Science and Innovative Technology*, 3(2), (ISSN : 2139 – 5967), 304 – 311, 2014*a*.

Chakrabarty Dhritikesh, Analysis of Errors Associated to Observations of Measurement Type, *International Journal of Electronics and Applied Research* (ISSN : 2395 – 0064), 1(1), 15 – 28, 2014*b*. Available at *http://eses.net.in/ESES Journal*.

Chakrabarty Dhritikesh, Observation Composed of a Parameter and Random Error: An Analytical Method of Determining the Parameter, *International Journal of Electronics and Applied Research* (ISSN : 2395 – 0064), 1(2), 20 – 38, 2014*c*. Available at *http://eses.net.in/ESES Journal*.

Chakrabarty Dhritikesh, Observation Consisting of Parameter and Error: Determination of Parameter, Proceedings of the World Congress on Engineering 2015, (WCE 2015, July 1 - 3, 2015, London, U.K.), ISBN: 978-988-14047-0-1, ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online), Vol. II, 680 – 684, 2015*a*.

Chakrabarty Dhritikesh: "Observation Consisting of Parameter and Error: Determination of Parameter," Lecture Notes in Engineering and Computer Science (ISBN: 978-988-14047-0-1), London, 680 – 684, 2015*b*. Available at https://www.researchgate.net/profile/Dhritikesh_Chakrabarty/stats.

Chakrabarty Dhritikesh, Central Tendency of Annual Extremum of Surface Air Temperature at Guwahati, *J. Chem. Bio. Phy. Sci.* (E- ISSN : 2249 – 1929), Sec. C, 5(3), 2863 – 2877, 2015*c*.

Chakrabarty Dhritikesh, Central Tendency of Annual Extremum of Surface Air Temperature at Guwahati Based on Midrange and Median, *J. Chem. Bio. Phy. Sci.* (E- ISSN : 2249 – 1929), Sec. D, 5(3), 3193 – 3204, 2015*d*.

Chakrabarty Dhritikesh, Observation Composed of a Parameter and Random Error: Determining the Parameter as Stable Range, *International Journal of Electronics and Applied Research* (ISSN : 2395 – 0064), 2(1), 35 – 47, 2015*e*.

Chakrabarty Dhritikesh, A Method of Finding Appropriate value of Parameter from Observation Containing Itself and Random Error, *Indian Journal of Scientific Research and Technology*, (E-ISSN: 2321-9262), 3(4), 14 – 21, 2015*f*.

Chakrabarty Dhritikesh, Theoretical Model Modified For Observed Data: Error Estimation Associated To Parameter, *International Journal of Electronics and Applied Research* (ISSN : 2395 – 0064), 2(2), 29 – 45, 2015*g*. Available at *http://eses.net.in/ESES Journal*.

Chakrabarty Dhritikesh, Impact of Error Contained in Observed Data on Theoretical Model: Study of Some Important Situations, *International Journal of Advanced Research in Science, Engineering and Technology*, (ISSN : 2350 – 0328), 3(1), 1255 – 1265, 2016*a*.

Chakrabarty Dhritikesh, Theoretical Model and Model Satisfied by Observed Data: One Pair of Related Variables, *International Journal of Advanced Research in Science, Engineering and Technology*, (ISSN : 2350 – 0328), 3(2), 1527 – 1534, 2016*b*.

Chakrabarty Dhritikesh, Pythagorean Mean: Concept behind the Averages and Lot of Measures of Characteristics of Data, NaSAEAST- 2016, *Abstract ID: CMAST_NaSAEAST (Inv)-1601)*, 2016*c*. Available at https://www.researchgate.net/profile/Dhritikesh_Chakrabarty/stats .

Chakrabarty Dhritikesh, Numerical Data Containing One Parameter and Random Error: Evaluation of the Parameter by Convergence of Statistic, *International Journal of Electronics and Applied Research* (ISSN : 2395 – 0064), 4(2), 59 – 73, 2017*a*. Available at *http://eses.net.in/ESES Journal*.

Chakrabarty Dhritikesh, Variable(s) Connected by Theoretical Model and Model for Respective Observed Data, 3rd International Conference on Fuzzy Systems and Data Mining (FSDM 2017), *Paper ID: FSDM2220*. 2017*b*. Available at https://www.researchgate.net/profile/Dhritikesh_Chakrabarty/stats .

Chakrabarty Dhritikesh, Objectives and Philosophy behind the Construction of Different Types of Measures of Average, NaSAEAST- 2017, *Abstract ID: CMAST_NaSAEAST (Inv)-1701)*, 2017*c*. Available at https://www.researchgate.net/profile/Dhritikesh_Chakrabarty/stats .

Chakrabarty Dhritikesh, Observed Data Containing One Parameter and Random Error: Evaluation of the Parameter Applying Pythagorean Mean, *International Journal of Electronics and Applied Research* (ISSN : 2395 – 0064), 5(1), 32 – 45, 2018*a*. Online (*http://eses.net.in/ESES Journal)*. Available at *http://eses.net.in/ESES Journal*.

Chakrabarty Dhritikesh, Derivation of Some Formulations of Average from One Technique of Construction of Mean, *American Journal of Mathematical and Computational Sciences*, 3(3), 62 – 68, 2018*b*.

Chakrabarty Dhritikesh, One Generalized Definition of Average: Derivation of Formulations of Various Means, *Journal of Environmental Science, Computer Science and Engineering & Technology*, Section C, (E-ISSN : 2278 – 179 X), 7(3), 212 – 225, 2018*c*.

Chakrabarty Dhritikesh, f_H - Mean: One Generalized Definition of Average, *Journal of Environmental Science, Computer Science and Engineering & Technology*, Section C, (E-ISSN : 2278 – 179 X), 7(4), 301 – 314, 2018*d*. Also available in www.jecet.org.

Chakrabarty Dhritikesh, General Technique of Defining Average, NaSAEAST- 2018, *Abstract ID: CMAST_NaSAEAST -1801 (I)*, 2018*e*. Available at https://www.researchgate.net/profile/Dhritikesh_Chakrabarty/stats .

Chakrabarty Dhritikesh, Generalized f_G - Mean: Derivation of Various Formulations of Average, *American Journal of Computation, Communication and Control*, 5(3), 101 – 108, 2018*f*. Available in http://www.aascit.org/journal/ajmcs .

Chakrabarty Dhritikesh, Significance of Change of Rainfall: Confidence Interval of Annual Total Rainfall, *Journal of Chemical, Biological and Physical Sciences* (E- ISSN : 2249 – 1929), Sec. C, 9(3), 151 – 166, 2019*a*. Available at: www.jcbsc.org.

Chakrabarty Dhritikesh, Observed Data Containing One Parameter and Random Error: Probabilistic Evaluation of Parameter by Pythagorean Mean, *International Journal of Electronics and Applied Research* (ISSN : 2395 – 0064), 6(1), 24 – 40, 2019*b*. Available at *http://eses.net.in/ESES Journal*.

Chakrabarty Dhritikesh, Significance of Change in Surface Air Temperature in the Context of India, *Journal of Chemical, Biological and Physical Sciences* (E- ISSN : 2249 – 1929), Sec. C, 9(4), 251 – 261, 2019*c*. Online available at: www.jcbsc.org .

Chakrabarty Dhritikesh, One General Method of Defining Average: Derivation of Definitions/Formulations of Various Means, *Journal of Environmental Science, Computer Science and Engineering & Technology*, Section C, (E-ISSN : 2278 – 179 X), 8(4), 327 – 338, 2019*d*. Also available in www.jecet.org .

Chakrabarty Dhritikesh, A General Method of Defining Average of Function of a Set of Values, *Aryabhatta Journal of Mathematics & Informatics* {ISSN (Print) : 0975-7139, ISSN (Online) : 2394-9309}, 11(2), 269 – 284, 2019*e*. Also available in www.abjni.com .

Chakrabarty Dhritikesh, Pythagorean Geometric Mean: Measure of Relative Change in a Group of Variables, NaSAEAST- 2019, *Abstract ID: CMAST_NaSAEAST -1902 (I)*, 2019*f*. Available at https://www.researchgate.net/profile/Dhritikesh_Chakrabarty/stats .

Chakrabarty Dhritikesh, Arithmetic-Geometric Mean: Evaluation of Parameter from Observed Data Containing Itself and Random Error, *International Journal of Electronics and Applied Research* (ISSN : 2395 – 0064), 6(2), 24 – 40, 2019*g*. Available at *http://eses.net.in/ESES Journal*.

Chakrabarty Dhritikesh, Definition / Formulation of Average from First Principle", *Journal of Environmental Science, Computer Science and Engineering & Technology*, Section C, (E-ISSN : 2278 – 179 X), 9(2), 151 – 163, 2020*a***.** Availableat www.jecet.org .

Chakrabarty Dhritikesh, AGM: A Technique of Determining the Value of Parameter from Observed Data Containing Itself and Random Error, *Journal of Environmental Science, Computer Science and Engineering & Technology*, Section C, (E-ISSN : 2278 – 179 X), 9(3), 473 – 486, 2020*b***.** Available at www.jecet.org .

Cornelli G., McKirahan R. & Macris C, *On Pythagoreanism*, Berlin, Walter de Gruyter, 2013.

de Carvalho Miguel, Mean, what do you Mean?, *The American Statistician*, 70, 764 –776, 2016.

David A. Cox , *The Arithmetic-Geometric Mean of Gauss, In J.L. Berggren; Jonathan M. Borwein; Peter Borwein (eds.). Pi: A Source Book. Springer. p. 481. ISBN 978-0-387-20571-7.* first published in *L'Enseignement Mathématique*, t. 30 (1984), p. 275-330, 2004

Hazewinkel, Michiel, ed. , *Arithmetic–geometric mean process, Encyclopedia of Mathematics, Springer Science+Business Media B.V. / Kluwer Academic Publishers, ISBN 978-1-55608-010-4,* 2001.

Ivory, On the Method of Least Squares, *Phil. M*ag., LXV, 3 – 10, 1825.

Kendall M. G. and Stuart A, Advanced Theory of Statistics, *Vol. 1 & 2, 4th Edition, New York, Hafner Press,* 1977.

Kolmogorov Andrey, On the Notion of Mean, *Mathematics and Mechanics*, 199(1), 144 – 146, 1930.

Lehmann Erich L. & Casella George, *Theory of Point Estimation, 2nd ed. Springer.* ISBN 0 – 387 – 98502 – 6, 1998.

Lucien Le Cam, Maximum likelihood — An introduction, *ISI Review*, 58 (2), 153 –171, 1990.

Michiel ed. Hazewinkel, Normal Distribution, *Encyclopedia of Mathematics*, Springer, ISBN 978 – 1 – 55608 – 010 – 4, 2001.

Walker Helen M. & Lev J., Statistical Inference, *Oxford & IBH Publishing Company*, 1965.

O'Meara Dominic J., *Pythagoras Revived: Mathematics and Philosophy in Late Antiquity*, ISBN 0-19-823913-0, Clarendon Press, Oxford, 1989.

Riedweg Christoph, Pythagoras: his life, teaching, and influence (translated by Steven Rendall in collaboration with Christoph Riedweg and Andreas Schatzmann, Ithaca), ISBN 0-8014-4240-0, *Cornell University Press*, 2005.

Weir, Alan J., *The Convergence Theorems, Lebesgue Integration and Measure. Cambridge: Cambridge University Press. pp. 93–118. ISBN 0-521-08728-7.*
1973.

Yeh, J., "*Real Analysis: Theory of Measure and Integration*", *Hackensack, NJ: World Scientific. ISBN 981-256-653-8*, 2006.

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