Recent Developments on Representation of Experimental Data by Polynomial Curve

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Abstract

Description / explanation of experimental data by mathematical curve / equation is one of the vital tools of analysis of data. Recently, some studies have been made on mathematical representation of experimental data on a pair of variables. In this context, a number of formulas based on mathematical operation like forward difference operation, backward difference operation, divided difference operation, backward divided difference operation, difference & ratio operation and backward difference & ratio operation for explaining the data on a pair of variables by suitable mathematical equation / curve. Moreover, some methods based on matrix inversion by Cayley-Hamilton Theorem, Gauss Jordan method and elementary column transformation have been derived for the same purpose in connection with the searching for some more meritorious formula / method of interpolation. A summative description of these formulas / methods has been discussed in this paper along with their applications in numerical data.

Keywords: Pair of variables, experimental data, mathematical representation, polynomial curve.

1. Introduction

Observations or data, collected from experiment or survey, normally suffer from various types of errors / causes which can be broadly divided into two types namely (1) Assignable error / cause that is avoidable / controllable & (2) Chance error / cause that is unavoidable / uncontrollable [Chakrabarty (2014a, 2014b, 2014c, 2014d, 2014e, 2014f, 2014g, 2014h, 2015a, 2015b, 2015c, 2015d, 2015e)]. Even if all the assignable causes of error are controlled or eliminated, observations still do not become free from error. Each of them still suffers from some error which occurs due to some unknown and unintentional cause.
that is nothing but the chance cause. Consequently the findings obtained by analyzing the observations which are free from the assignable errors are also subject to errors due to the presence of chance error in the observations. Determination of constant(s) associated to mathematical model(s), in different situations, based on the observations is also subject to error due to the same reason.

A number of mathematical models have been identified for describing the association of chance error(s) in determining constant(s) in some distinct situations where observations/data are of measurement type [Chakrabarty (2014f, 2016a, 2016b, 2017e)].

There are two broad aspects of statistical determination of parameters involved in the respective models describing the dependence of the dependent variable on the independent variable(s). One of them is based on the basic philosophy behind statistics [Chakrabarty (2018a, 2018g, 2019c)], which consists of determining the parameter(s) from numerical data compromising with some degree of error in findings. Several statistical methods have already been developed for determination (estimation in statistical literature) of such parameter(s) which are available in the standard literatures in statistics. However, existing statistical methods of estimation cannot normally yield error free estimate(s) of parameters [Chakrabarty (2014f, 2016a, 2016b)]. The same fact happens in the case of some recently developed methods of estimation [Chakrabarty (2011, 2014d, 2016c, 2016d, 2016e), Chakrabarty & Dutta (2007), Chakrabarty & Rahman (2007, 2008)]. Recently, some studies have been made on attempting of determining error free estimates of such parameter(s) [Chakrabarty (2014b, 2014f, 2014h, 2015a, 2015d, 2015e, 2017f, 2017g, 2018d, 2019d, 2019i, 2020b)] along with attempting of application in the case of real data [Chakrabarty (2015c, 2015-16, 2019b, 2019e)]. These studies have been done on the basis of various measures of average namely the three Pythagorean means (namely arithmetic mean, geometric mean & harmonic mean), median, generalized mean and others [Chakrabarty (2016h, 2017f, 2018b, 2018c, 2018d, 2018e, 2018f, 2019a, 2019f, 2019g, 2019h, 2020a)].

The recent trend is towards the study on the representation of numerical data on a pair of variables by suitable mathematical equation / mathematical curve [Chakrabarty (2016e , 2016f , 2016g , 2016h , 2016i , 2016 – 17, 2017b , 2017c , 2018h), Das & Chakrabarty (2016a , 2016b , 2016c , 2016d , 2016e , 2016f , 2017a 2017b , 2017c , 2017d)] in connection with the development of some more meritorious formula / method of interpolation which is a technique of estimating approximately the value of the dependent variable corresponding to a value of the independent variable lying between its two extreme values on the basis of the given values of the independent and the dependent variables [Hummel (1947), Erdos Turan (1938) , Bathe & Wilson (1976) , Jan (1930) , Hummel (1947) et al].

The following formulas / methods have been developed in this studies:

1. One formula (Das & Chakrabarty, 2016a) based on usual algebraic operation.
2. One formula (Das & Chakrabarty, 2016c) based on forward difference operation.
3. One formula (Das & Chakrabarty, 2016d) based on backward difference operation.
4. One formula (Das & Chakrabarty, 2016b) based on divided difference operation.
5. One method (Das & Chakrabarty, 2016e) based on matrix inversion by Cayley-Hamilton Theorem [Cayley (1858, 1889) & Hamilton (1864a, 1864b, 1862)].
6. One method (Das & Chakrabarty, 2016f) based on matrix inversion by Gauss Jordan method [Grcar & Joseph (2011a, 2011b); Kaw, Autar, Kalu & Egwu (2010)] which is based on elementary row transformation.
7. One method (Das & Chakrabarty, 2017a) based on matrix inversion by elementary column transformation.
8. One formula (Das & Chakrabarty, 2017b) based on backward divided difference operation.
9. One formula (Chakrabarty, 2016g) based on difference and ratio operation.
10. One formula (Chakrabarty, 2016 – 17) based on backward difference and ratio operation.

The first eight formulas / methods are the basic ones.

A summative description of these formulas / methods has been discussed in this paper along with their applications in numerical data.

2. Formula / Method Representation of Numerical Data
Formulas / methods of representation of numerical data on a pair of variables by suitable mathematical equation / mathematical curve, as mentioned above, have been discussed below:

2.1. **Forward Difference Formula:**

Let us consider the situation where the argument $x$ assume the values which are equally spaced i.e. $x$ assume the values at equal interval and let the length of the interval be $h$.

Thus,

$$x_{i+1} - x_i = h, \quad (i = 0, 1, 2, \ldots, n - 1)$$

Let us define a variable $u$ by

$$u = \frac{x-x_0}{h}$$

Then the forward difference formula for representing a given set of numerical data on a pair of variables by a suitable polynomial curve in the situation where the argument assumes the values which are equally spaced i.e. the argument assumes the values at equal interval is given by

$$y = f(x) = \alpha_0 x^0 + \alpha_1 x^1 + \alpha_2 x^2 + \ldots \ldots + \alpha_n x^n \quad (I)$$

where

$$\alpha_0 = A_0 - A_1 x_0 + A_2 x_0 x_1 - A_3 x_0 x_1 x_2 + A_4 x_0 x_1 x_2 x_3 - \ldots$$

$$\ldots$$

$$+ A_n (-1)^n (x_0 x_1 x_2 x_3 \ldots \ldots x_{n-1}) ,$$

$$\alpha_1 = A_1 - A_2 (\sum_{i=0}^{1} x_i) + A_3 (\sum_{i=0}^{2} \sum_{j=1}^{1} x_i x_j) -$$

$$+ A_4 (\sum_{i=0}^{1} \sum_{j=1}^{2} \sum_{k=2}^{2} x_i x_j x_k) + \ldots$$

$$+ \ldots$$

$$+ (-1)^n A_n (x_0 x_1 x_2 x_3 \ldots \ldots x_{n-2}$$

$$+ x_0 x_1 x_2 x_3 \ldots \ldots x_{n-1}) ,$$

$$\alpha_2 = A_2 - A_3 (\sum_{i=0}^{2} x_i) + A_4 (\sum_{i=0}^{2} \sum_{j=1}^{1} x_i x_j) -$$

$$+ (-1)^n A_n (x_0 x_1 x_2 x_3 \ldots \ldots x_{n-3} + x_0 x_1 x_2 x_3 \ldots \ldots x_{n-2}$$

$$+ x_0 x_1 x_2 x_3 \ldots \ldots x_{n-1}) ,$$

$$\alpha_3 = A_3 - A_4 (\sum_{i=0}^{3} x_i) + \ldots$$

$$+ (-1)^n A_n (x_0 x_1 x_2 x_3 \ldots \ldots x_{n-4}$$

$$+ x_0 x_1 x_2 x_3 \ldots \ldots x_{n-3} + x_0 x_1 x_2 x_3 \ldots \ldots x_{n-2} + x_0 x_1 x_2 x_3$$

$$\ldots$$

$$+ x_0 x_1 x_2 x_3 \ldots \ldots x_{n-1}) ,$$

$$\ldots$$

$$\ldots$$
\[ \alpha_i = A_i - A_{i+1} (\sum^i_{j=0} x_j) + \ldots + (-1)^{n-i} A_n (x_0x_1x_2x_3 \ldots \ldots \ldots \ldots x_{n-1}) \]

\[ \alpha_i = A_n \]

with

\[ A_0 = y_0, \ A_1 = \frac{y_0}{h}, \ A_2 = \frac{y_0}{2h^2}, \ A_i = \frac{\Delta y_0}{(i-1)!h^{i-1}}, \ldots, A_n = \frac{\Delta^n y_0}{n!h^n} \]

(Das & Chakrabarty, 2016c):

### 2.1. Backward Difference Formula

As in the earlier case, here also let us consider the situation where the argument \( x \) assume the values which are equally spaced i.e. \( x \) assume the values at equal interval and let the length of the interval be \( h \). Thus,

\[ x_{i+1} - x_i = h \quad (i = 0, 1, 2, \ldots , n-1) \]

Let us define a variable \( v \) by

\[ v = \frac{x-x_n}{h} \]

Then the backward difference formula for representing a given set of numerical data on a pair of variables by a suitable polynomial curve in the situation where the argument assumes the values which are equally spaced i.e. the argument assumes the values at equal interval is given by

\[ y = f(x) = \beta_0 x^0 + \beta_1 x^1 + \beta_2 x^2 + \ldots \ldots + \beta_n x^n \quad (2) \]

where

\[ \beta_0 = B_n - B_{n-1} x_n + B_{n-2} x_n x_{n-1} - B_{n-3} x_{n-2} x_{n-1} x_n + \]

\[ B_{n-4} x_{n-3} x_{n-2} x_{n-1} x_n - \ldots (-1)^{n} B_0 (x_n x_{n-1} x_{n-2} x_{n-3} \ldots \ldots x_1) \]

\[ \beta_1 = B_{n-1} - B_{n-2} (\sum^n_{i=n-1} x_i) + B_{n-3} (\sum^{n-1}_{i=n-2} \sum^{n}_{j=i} x_i x_j) - B_{n-4} \]

\[ (\sum^{n-2}_{i=n-3} \sum^{n-1}_{j=n-2} \sum^{n}_{k=n-1} x_i x_j x_k) + \ldots \ldots - B_0 (\sum^3_{i=1} \sum^1_{j=2} \sum^0_{k=1} x_i x_j x_k) \]

\[ \beta_2 = B_{n-2} - B_{n-3} (\sum^n_{i=n-2} x_i) + B_{n-4} (\sum^n_{i=n-3} \sum^{n}_{j=n-2} x_i x_j) - \ldots \ldots + \]

\[ B_0 (\sum^3_{i=2} \sum^2_{j=3} \sum^1_{k=2} \sum^0_{l=1} x_i x_j x_k x_l) \]
\[ \beta_3 = B_{n-3} - B_{n-4}(\Sigma_{i=n-3}^{n} x_i) + \ldots + \]
\[ B_0 \ (\Sigma_{i=3}^{4} \Sigma_{j=4}^{2} \Sigma_{k=3}^{1} \Sigma_{m=1}^{0} x_i x_j x_k x_m) \]

\[ \beta_n = B_n \]

with
\[ B_n = f(x_n), \ B_{n-1} = \frac{\nu f(x_{n-1})}{h}, \ B_{n-2} = \frac{\nu^2 f(x_{n-2})}{2h^2}, \ B_{n-3} = \frac{\nu^3 f(x_{n-3})}{3h^3}, \]
\[ B_{n-4} = \frac{\nu^4 f(x_{n-4})}{4h^4}, \ldots \]
\[ B_0 = \frac{\nu^n f(x_0)}{n!h^n} \]

(Das & Chakrabarty, 2016d).

### 2.3. Divided Difference Formula:

Let us now consider the situation where the argument \( x \) assume the values which are not equally spaced i.e. \( x \) assume the values not at equal interval.

Then the divided difference formula, derived from Newton’s divided difference interpolation formula [Chwaiger (1994), De Boor (2003), Gertrude (1954) et al], for representing a given set of numerical data on a pair of variables by a polynomial curve in the situation where the argument assume the values which are not necessarily equally spaced is given by

\[ y = f(x) = \gamma_0 x^0 + \gamma_1 x^1 + \gamma_2 x^2 + \ldots + \gamma_n x^n \] (3)

where
\[ \gamma_0 = D_0 - D_1 x_0 + D_2 x_0 x_1 - D_3 x_0 x_1 x_2 + D_4 x_0 x_1 x_2 x_3 - \]
\[ \ldots + D_n(-1)^n \Pi_{i=0}^{n-1} x_i \]

\[ \gamma_1 = D_1 - D_2 (\Sigma_{i=0}^{1} x_i) + D_3 (\Sigma_{i=0}^{1} \Sigma_{j=1}^{2} x_i x_j) - D_4 (\Sigma_{i=0}^{1} \Sigma_{j=1}^{2} \Sigma_{k=2}^{3} x_i x_j x_k) + \]
\[ \ldots +(-1)^n D_n (\Pi_{i=0}^{n-2} x_i + \Pi_{i=0}^{n-1} x_i) \]

\[ \gamma_2 = D_2 - D_3 (\Sigma_{i=0}^{2} x_i) + D_4 (\Sigma_{i=0}^{2} \Sigma_{j=1}^{3} x_i x_j) - \ldots - D_n (\Pi_{i=0}^{n-3} x_i + \Pi_{i=0}^{n-2} x_i + \Pi_{i=0}^{n-1} x_i) \]

\[ \gamma_3 = D_3 - D_4 (\Sigma_{i=0}^{3} x_i) + \ldots +(-1)^n D_n (\Pi_{i=0}^{n-4} x_i + \Pi_{i=0}^{n-3} x_i + \Pi_{i=0}^{n-2} x_i + \Pi_{i=0}^{n-1} x_i) \]

\[ \gamma_n = D_n - D_{n+1} (\Sigma_{i=0}^{n} x_i) + \ldots +(-1)^{n-i} D_n (\Pi_{i=0}^{n-i} x_i) \]
\[ y = \rho_0 x^0 + \rho_1 x^1 + \rho_2 x^2 + \ldots + \rho_n x^n \]  

(4)

where

\[ \rho_0 = \sum_{i=0}^{n} L_i S_i(n) \]  
\[ \rho_1 = \sum_{i=0}^{n} L_i S_i(n - r) \]  
\[ \rho_2 = \sum_{i=0}^{n} L_i S_i(2) \]  
\[ \rho_3 = \sum_{i=0}^{n} L_i S_i(3) \]  
..........................  
\[ \rho_{n-r} = \sum_{i=0}^{n} L_i S_i(r) \]  
..........................

2.4. Algebraic Formula:

Let the values of the independent variable \( x \) be not necessary at equal interval. Then the algebraic formula, obtained from Lagrange’s interpolation formula [Whittaker & Robinson (1967), Echols (1893), Mills (1977) et al], for representing a given set of numerical data on a pair of variables by a polynomial curve in the situation where the argument assume the values which are not necessarily equally spaced is given by

\[ y = f(x) = \rho_0 x^0 + \rho_1 x^1 + \rho_2 x^2 + \ldots + \rho_n x^n \]  

(4)
\[
\begin{align*}
\rho_{n-2} &= \sum_{i=0}^{n} L_i S_i(2), \\
\rho_{n-1} &= S_i(1), \\
\rho_n &= \sum_{i=0}^{n} L_i \\
\text{with} \\
L_0 &= \frac{f(x_0)}{\prod_{i=0}^{n} (x_0 - x_i)}, \\
L_1 &= \frac{f(x_1)}{\prod_{i=0}^{n} (x_1 - x_i)} \\
&\quad i \neq 0, i \neq 1 \\
\ldots \ldots \ldots \ldots \\
L_n &= \frac{f(x_n)}{\prod_{i=0}^{n-1} (x_n - x_i)} \\
&\quad i \neq n-1
\end{align*}
\]

and

\[
S_r = \sum_{i=0}^{n} x_i, \quad S_r (p) = \sum_{i=0}^{n} \sum_{j=0}^{n} x_i x_j \\
i \neq r \\
i \neq r, j \neq r, i \neq j
\]

\[
S_{r_1 r_2 \ldots \ldots r_p} = \sum_{i_1=0}^{n} \sum_{i_2=0}^{n} \ldots \ldots \sum_{i_p=0}^{n} x_{i_1} x_{i_2} \ldots \ldots x_{i_p} \\
i_1 \neq r_1, i_2 \neq r_1 \ldots \ldots i_p \neq r_p \\
i_1 \neq i_2 \neq \ldots \ldots \neq i_p
\]

(Das & Chakrabarty, 2016a).

2.5. Backward Divided Difference Formula:

As earlier let the values of the independent variable \(x\) be not necessary at equal interval.

Then the backward divided difference formula for representing a given set of numerical data on a pair of variables by a polynomial curve in the situation where the argument assume the values which are not necessarily equally spaced is given by

\[
y = f(x) = \delta_0 x^0 + \delta_1 x^1 + \delta_2 x^2 + \ldots \ldots + \delta_n x^n
\]
where

\[
\delta_n = E_n - E_{n-1}x_n + E_{n-2}x_{n-1}x_n - E_{n-3}x_{n-2}x_{n-1}x_n + \\
+ \cdots + (-1)^{n}E_0(x_{n-1}x_{n-2}x_{n-3} \cdots x_1 x_0) \\
\delta_{n-1} = E_{n-1} - E_{n-2}(\sum_{i=n-1}^{n} x_i) + E_{n-3}(\sum_{i=n-2}^{n-1} \sum_{j=n-1}^{n-2} x_i x_j) - \\
+ \cdots + (-1)^{n}C_0(x_n x_{n-1} x_{n-2} x_{n-3} \cdots x_1 x_0) \\
\delta_{n-2} = E_{n-2} - E_{n-3}(\sum_{i=n-2}^{n} x_i) + E_{n-4}(\sum_{i=n-3}^{n-2} \sum_{j=n-2}^{n-3} x_i x_j) - \\
+ \cdots + (-1)^{n}E_0(x_n x_{n-1} x_{n-2} x_{n-3} \cdots x_3 + \\
x_n x_{n-1} x_{n-2} x_{n-3} \cdots x_2 + x_n x_{n-1} x_{n-2} x_{n-3} \cdots x_1) \\
\delta_{n-3} = E_{n-3} - E_{n-4}(\sum_{i=n-3}^{n} x_i) + \cdots + (-1)^{n}E_0(x_n x_{n-1} x_{n-2} x_{n-3} \\
\vdots x_4 + x_n x_{n-1} x_{n-2} x_{n-3} \cdots x_3 + x_n x_{n-1} x_{n-2} x_{n-3} \cdots x_2 \\
+ x_n x_{n-1} x_{n-2} \cdots x_1) \\
\vdots \\
\delta_{n-i} = E_{n-i} - E_{n-(i+1)}(\sum_{i=n-1}^{n} x_i) + \cdots + (-1)^{n-i}E_0 \\
(x_n x_{n-1} x_{n-2} \cdots x_{n-(i+1)} + \cdots + x_n x_{n-1} x_{n-2} \cdots x_1) \\
\delta_0 = E_0 \\
\]

with

\[
E_n = f(x_n) \\
E_{n-1} = f(x_n, x_{n-1}) \\
E_{n-2} = f(x_n, x_{n-1}, x_{n-2}) \\
E_{n-3} = f(x_n, x_{n-1}, x_{n-2}, x_{n-3}) \\
\vdots \\
E_{n-i} = (x_n, x_{n-1}, x_{n-2}, x_{n-3}, \ldots, x_{n-i}) \\
\vdots \\
E_0 = f(x_n, x_{n-1}, x_{n-2}, x_{n-3}, \ldots, x_0) \\
\]

(Das & Chakrabarty, 2017b).
2.6. Method of Representation by Matrix Inversion using Characteristic Equation:

Suppose, \( y_i \) denotes the value of the function (also called entry)

\[
y = f(x)
\]

corresponding to the value \( x_i \) of the independent variable (also called argument) \( x \)
where \( i \) assumes the integral values from 0 to \( n \).

The problem here is to represent the \((n + 1)\) pairs of values namely

\((x_i, y_i)\), for all \( i \),

by the polynomial curve of the form

\[
y = f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n, \quad a_n \neq 0
\]  \( (6) \)

Since the \((n + 1)\) points lie on the curve describe by equation \((7)\),

\[
y_i = a_0 + a_1 x_i + a_2 x_i^2 + \ldots + a_n x_i^n, \text{ for all } i
\]  \( (7) \)

Solving these equations for the parameters, the form of the polynomial curve can be obtained which can represent the values of the \((n + 1)\) pairs of values of the argument and entry.

The \((n + 1)\) equations given by \((7)\) can be expressed as

\[
A X = B
\]  \( (8) \)

where

(i) A is a square matrix of order \((n + 1)\) given by

\[
A = \begin{pmatrix}
1 & x_0 & x_0^2 & \ldots & x_0^n \\
1 & x_1 & x_1^2 & \ldots & x_1^n \\
& \vdots & \ddots & \vdots & \vdots \\
1 & x_n & x_n^2 & \ldots & x_n^n
\end{pmatrix}
\]

(ii) \( X \) is a column vector of order \((n + 1)\) given by

\[
X = \begin{pmatrix}
a_0 \\
a_1 \\
a_2 \\
\vdots \\
a_n
\end{pmatrix}
\]
& (iii) $B$ is a column vector of order $(n+1)$ given by

$$B = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ . \\ . \\ . \\ y_n \end{pmatrix}$$

From equation (9), we get

$$X = A^{-1}B$$

This gives the values of the coefficients of the polynomial curve given by equation (6).

Thus, in order to find out the values of the above coefficients, it is essential to find out the inverse of the matrix $A$.

For finding out the inverse of the matrix $A$, one can apply the Cayley-Hamilton theorem [Cayley (1858), Cayley (1889), Hummel (1947), Hamilton (1864a) et al] on the characteristic equation of $A$ given by

$$|A - \lambda \cdot I_{n+1}| = 0 \quad (9)$$

where $\lambda$ is the characteristic root (also known as the eigen value or the spectral value) of the matrix $A$ and $I_{n+1}$ is the identity matrix of order $(n+1)$.

Cayley-Hamilton theorem implies that every square matrix satisfies its own characteristic equation. Thus,

$$|A - A \cdot I_{n+1}| = 0 \quad (10)$$

If the characteristic equation of the matrix $A$ is given by equation (6) becomes of the form

$$\lambda^n - c_1 \lambda^{n-1} + c_2 \lambda^{n-2} - c_3 \lambda^{n-3} + \ldots + (-1)^n c_n = 0 \quad (11)$$

after algebraic expansion, then by Cayley-Hamilton theorem the matrix $A$ will be the solution of $\lambda$. i.e.

$$\lambda = A$$

Hence from equation (12) it can be obtained that

$$A^{-1} = \frac{1}{(-1)^n c_n} \left[ A^n - c_1 A^{n-1} + c_2 A^{n-2} - c_3 A^{n-3} + \ldots + (-1)^{n-1} c_n I_{n+1} \right] \quad (12)$$

[Cayley (1858, 1889), Hummel (1947, 1864a), Das & Chakrabarty (2016e)].
2.7. Method of Representation by Inversion of Matrix using Elementary Row Transformation:

In order to represent the \((n + 1)\) pairs of values namely 
\((x_i, y_i)\), for all \(i\), by the polynomial curve described by equation (6), it is necessary to solve the equations (7) for which it is again necessary to find out the inverse of the matrix \(A\) given by

\[
\begin{pmatrix}
1 & x_0 & x_0^2 & ... & x_0^n \\
1 & x_1 & x_1^2 & ... & x_1^n \\
... & ... & ... & ... & ... \\
1 & x_n & x_n^2 & ... & x_n^n \\
\end{pmatrix}
\]

The inverse of the matrix \(A\) can also be found out by Gauss Jordan method [David (1991), Endre (2003) et al] which is based on elementary row transformations of matrix.

Working rule:

The working steps of finding inverse are as follows:

(i) Apply elementary row transformations on the matrix \(A\) to obtain the identity matrix.

(ii) Apply the same elementary row transformations in the same order on the identity matrix.

(iii) The transformed matrix obtained from the identity matrix is the required inverse matrix of the matrix \(A\).

Thus in order to find the inverse of \(A\) by E-row operations, we write \(A\) and the identity matrix \(I\) side by side and the same operations are performed on both. As soon as the matrix \(A\) is reduced to the identity matrix \(I\), the identity matrix \(I\) will reduce to \(A^{-1}\) [David (1991), Endre (2003), Das & Chakrabarty (2016f)].

2.8. Method of Representation by Inversion of Matrix using Elementary Column Transformation:

In order to represent the \((n + 1)\) pairs of values namely 
\((x_i, y_i)\), for all \(i\),
by the polynomial curve described by equation (6), it is necessary to solve the equations (8) for which it is again necessary to find out the inverse of the matrix \( A \) given by

\[
\begin{pmatrix}
1 & x_0 & x_0^2 & \ldots & x_0^n \\
1 & x_1 & x_1^2 & \ldots & x_1^n \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & x_n & x_n^2 & \ldots & x_n^n
\end{pmatrix},
\]

**Working rule:**

The working steps of finding inverse are as follows:

(i) Apply elementary column transformations on the matrix \( A \) to obtain the identity matrix.

(ii) Apply the same elementary column transformations in the same order on the identity matrix.

(iii) The transformed matrix obtained from the identity matrix is the required inverse matrix of the matrix \( A \).

Thus in order to find the inverse of \( A \) by E-column operations, we write \( A \) and the identity matrix \( I \) side by side and the same operations are performed on both. As soon as the matrix \( A \) is reduced to the identity matrix \( I \), the identity matrix \( I \) will reduce to \( A^{-1} \) [David (1991), Endre (2003), Das & Chakrabarty (2017a)].

**Application to Numerical Data**

If the following table shows the number of persons (in thousands) in the population of India:

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
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<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of persons (in thousands)</td>
<td>548159.652</td>
<td>683329.097</td>
<td>846302.688</td>
<td>1027015.247</td>
<td>1210193.422</td>
</tr>
</tbody>
</table>

Taking 1971 as origin and changing scale by 1/10, one can obtain the following table for the values of the argument \( x \) (representing year) and the entry \( y = f(x) \) (representing the number of persons in the population of India):
Table – 3(ii)

(Values of argument and entry of Number of persons in the population of India)

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Value of $x_i$</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>Value of $y_i = f(x_i)$ (in thousands)</td>
<td>548159.652</td>
<td>683329.097</td>
<td>846302.688</td>
<td>1027015.247</td>
<td>1210193.422</td>
</tr>
</tbody>
</table>

Here, $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $x_3 = 3$, $x_4 = 4$

$f(x_0) = 548159.652$, $f(x_1) = 683329.097$, $f(x_2) = 846302.688$,

$f(x_3) = 1027015.247$, $f(x_4) = 1210193.422$

3.1. Representation by Forward Difference Formula:

Applying the equations given in (2) it is found that

$A_0 = 548159.652$,

$A_1 = 135169.445$,

$A_2 = 13902.073$,

$A_3 = -1677.529$,

$A_4 = -217.007$

Accordingly,

$\alpha_0 = 548159.652$,

$\alpha_1 = 119214.356$,

$\alpha_2 = 16547.583$,

$\alpha_3 = -375.487$,

$\alpha_4 = -217.007$

Consequently, the equation of the polynomial that can represent the given numerical data becomes

$f(x) = 548159.652 + 119214.356x + 16547.583x^2 - 375.487x^3 - 217.007x^4$ $\quad (16)$

3.2. Representation by Backward Difference Formula:
Applying the equations given in (2) we obtain the following:

\[ B_4 = 1210193.422, \]
\[ B_3 = 183178.175, \]
\[ B_2 = 1232.808, \]
\[ B_1 = -2545.55866, \]
\[ B_0 = -217.00725, \]

Accordingly, the following have been obtained:

\[ \beta_0 = 548159.652, \]
\[ \beta_1 = 119214.35634, \]
\[ \beta_2 = 16547.58219, \]
\[ \beta_3 = -375.48616, \]
\[ \beta_4 = -217.00725 \]

Consequently, the polynomial that can represent the given numerical data becomes

\[ f(x) = 548159.652 + 119214.356x + 16547.583x^2 - 375.487x^3 - 217.007x^4 \]  \hspace{1cm} (17)

### 3.3. Representation by Divided Difference Formula:

Applying the equations given in (3) we obtain the following:

\[ D_0 = 548159.652, \]
\[ D_1 = 135169.445, \]
\[ D_2 = 13902.073, \]
\[ D_3 = -1677.52966, \]
\[ \& \ D_4 = -217.00725 \]

Accordingly, the following have been obtained:

\[ \gamma_0 = 548159.652, \]
\[ \gamma_1 = 119214.35618, \]
\[ \gamma_2 = 16547.58223, \]
\[ \gamma_3 = -375.48616, \]
\[ \& \ \gamma_4 = -217.00725 \]
Consequently, the equation of the polynomial that can represent the given numerical data becomes

\[ f(x) = 548159.652 + 119214.356x + 16547.583x^2 - 375.487x^3 - 217.007x^4 \]  \hspace{1cm} (18)

### 3.4. Representation by Algebraic Formula:

Applying the formulas of \( L_i \), we obtain the following:

\[
\begin{align*}
L_0 &= 22839.9855, \\
L_1 &= -113888.18283, \\
L_2 &= 211575.672, \\
L_3 &= -171169.20783, \\
& \quad \& L_4 = 50424.72591.
\end{align*}
\]

Applying the formulas of \( \rho_i \), we obtain the following:

\[
\begin{align*}
\rho_0 &= 548159.652, \\
\rho_1 &= -119214.356, \\
\rho_2 &= 16547.5823, \\
\rho_3 &= 375.4862, \\
& \quad \& \rho_4 = -217.00725
\end{align*}
\]

Therefore, the polynomial that can represent the given numerical data becomes

\[ f(x) = -217.00725x^4 - 375.4862x^3 + 16547.5823x^2 + 119214.356x + 548159.652 \]

### 3.5. Representation by Backward Divided Difference Formula:

Applying the equations given in (5) we obtain the following:

\[
\begin{align*}
E_4 &= 1210193.422, \\
E_3 &= -183178.175, \\
E_2 &= 1232.808, \\
E_1 &= 2545.55866, \\
& \quad \& E_0 = -217.00725
\end{align*}
\]

Accordingly, the following have been obtained:
\[ \delta_1 = 1210193.422, \]
\[ \delta_3 = -178017.82218, \]
\[ \delta_2 = -8790.94773, \]
\[ \delta_1 = 3847.60216, \]
\[ \delta_0 = -217.00725 \]

Consequently, the equation of the polynomial that can represent the given numerical data becomes
\[ f(x) = 1210193.422 - 178017.82218 x - 8790.94773 x^2 + 3847.60216 x^3 - 217.00725 x^4 \]

3.6. Representation by Cayley-Hamilton Theorem:

Here
\[ X = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{pmatrix} & B = \begin{pmatrix} 548159.652 \\ 683329.097 \\ 846302.688 \\ 1027015.247 \end{pmatrix} \]

In order to obtain the characteristic equation of \( A \), it is necessary to expand the characteristic function \([A - \lambda I]\).

Its algebraic expansion is
\[ \lambda^4 - 33\lambda^3 + 94\lambda^2 - 74\lambda + 12 \]

Therefore the characteristic equation of \( A \) becomes
\[ |A - \lambda I| = 0 \]
\[ i.e. \lambda^4 - 33\lambda^3 + 94\lambda^2 - 74\lambda + 12 = 0 \]

Application of Cayley Hamilton’s theorem [Cayley (1858), Cayley (1889), Hummel (1947), Hamilton (1864a) et al] yield that
\[ A^{-1} = \frac{1}{12} [-A^3 + 33 A^2 - 94 A + 74 I ] \quad (18) \]

Now, numerical values \( A^2 \& A^3 \) are as follows:
\[ A^2 = A.A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 4 & 6 & 14 & 36 \\ 15 & 34 & 90 & 250 \\ 40 & 102 & 282 & 804 \end{pmatrix} \]
\[ A^3 = A^2 \cdot A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 60 & 142 & 386 & 1090 \\ 389 & 964 & 2644 & 7504 \\ 1228 & 3078 & 8466 & 24066 \end{pmatrix} \]

Therefore,
\[
A^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{-11}{6} & 3 & \frac{-3}{2} & \frac{1}{3} \\ 1 & \frac{-5}{2} & 2 & \frac{-1}{2} \\ \frac{-1}{6} & \frac{1}{2} & \frac{-1}{2} & \frac{1}{6} \end{pmatrix}
\]

Consequently, from \( AX = B \) it has been obtained that
\[
X = A^{-1}B = \begin{pmatrix} 548159.652 \\ 117912.31266 \\ 18934.662 \\ -1677.52967 \end{pmatrix}
\]

Thus the polynomial curve representing the data becomes
\[
y = f(x) = 548159.652 + 117912.31266 x + 18934.662 x^2 - 1677.52967 x^3 \quad (19)
\]

### 3.7. Representation by elementary row transformation of matrix:

As earlier here also,
\[
X = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{pmatrix} \quad \& \quad B = \begin{pmatrix} 548159.652 \\ 683329.097 \\ 846302.688 \\ 1027015.247 \end{pmatrix}
\]

Applying the following order elementary row transformations:

\[ R_2 \rightarrow R_2 - R_1, \quad R_3 \rightarrow R_3 - R_1, \quad R_4 \rightarrow R_4 - R_1, \quad R_3 \rightarrow \frac{1}{2} R_3, \quad R_3 \rightarrow R_3 - R_2, \quad R_4 \rightarrow \frac{1}{3} R_4, \quad R_4 \rightarrow R_4 - R_2 \]

\[ R_4 \rightarrow \frac{1}{2} R_4, \quad R_4 \rightarrow R_4 - R_3, \quad R_3 \rightarrow \frac{1}{3} R_3, \quad R_3 \rightarrow R_4 - R_3, \quad R_2 \rightarrow R_2 - R_4, \quad R_3 \rightarrow 3R_3 \]

\[ R_2 \rightarrow R_2 - R_3, \]

on the matrix \( A \) serially in the order, the matrix \( A \) has been converted to the identity matrix.
Now, on the applications of the same elementary raw transformations serially in the same order on the identity matrix of order 4, the identity matrix has been converted into the matrix

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
11 & 0 & 0 & 0 \\
-\frac{11}{6} & 3 & -\frac{3}{2} & \frac{1}{3} \\
1 & -\frac{5}{2} & 2 & -\frac{1}{2} \\
-\frac{1}{2} & 1 & 1 & 1 \\
-\frac{6}{2} & -\frac{2}{2} & -\frac{1}{6}
\end{pmatrix}
\]

Hence,

\[
A^{-1} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
-\frac{11}{6} & 3 & -\frac{3}{2} & \frac{1}{3} \\
1 & -\frac{5}{2} & 2 & -\frac{1}{2} \\
-\frac{1}{2} & 1 & 1 & 1 \\
-\frac{6}{2} & -\frac{2}{2} & -\frac{1}{6}
\end{pmatrix}
\]

Consequently, from

\[AX = B\]

it has been obtained that

\[
X = A^{-1}B = \begin{pmatrix}
548159.652 \\
117912.31266 \\
18934.662 \\
-1677.52967
\end{pmatrix}
\]

Thus the polynomial curve representing the data becomes

\[y = f(x) = 548159.652 + 117912.31266 \times x + 18934.662 \times x^2 - 1677.52967 \times x^3\]  

(20)

3.8. Representation by elementary column transformation of matrix:

As earlier here also,

\[X = \begin{pmatrix}
a_0 \\
a_1 \\
a_2 \\
a_3
\end{pmatrix}, \quad A = \begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 \\
1 & 2 & 4 & 8 \\
1 & 3 & 9 & 27
\end{pmatrix}, \quad B = \begin{pmatrix}
548159.652 \\
683329.097 \\
846302.688 \\
1027015.247
\end{pmatrix}\]

Applying the following order of elementary column transformations:
C₄ → C₄ – C₃ ,  C₃ → C₃ – C₂ ,  C₄ → C₄ – C₃ ,  C₂ → C₂ – C₄ ,  C₃ → C₃ – C₄ ,
C₂ → C₂ – C₃ ,  C₂ → C₂ – C₃ ,  C₁ → C₁ – C₂ ,  C₁ → C₁ – C₃ ,  C₄ → C₄ – C₃ ,  C₁ → C₁ – C₄ 

on the matrix A serially in the order, the matrix A has been converted to the identity matrix.

Now, on the applications of the same elementary column transformations serially in the same order on the
identity matrix of order 4, the identity matrix has been converted into the matrix

\[
A^{-1} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
\frac{11}{6} & \frac{3}{2} & \frac{1}{3} \\
1 & -2 & \frac{1}{2} \\
-\frac{1}{6} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{6}
\end{pmatrix}
\]

Hence

Consequently, from

\[AX = B\]

it has been obtained that

\[
X = A^{-1}B = \begin{pmatrix}
548159.652 \\
117912.31266 \\
18934.662 \\
-1677.52967
\end{pmatrix}
\]

Thus the polynomial curve representing the data becomes

\[y = f(x) = 548159.652 + 117912.31266 x + 18934.662 x^2 - 1677.52967 x^3\]
4. Conclusion:

Each of the formulas / methods yields the values of the function $f(x)$ corresponding to the respective observed values as follows:

$$
\begin{align*}
    f(0) &= 548159.652, \\
    f(1) &= 683329.097, \\
    f(2) &= 846302.688, \\
    \&f(3) &= 1027015.247.
\end{align*}
$$

Consequently, the Number of persons (in thousand) the population of India as yielded by the curves as follows:

\begin{align*}
    \text{Number of persons in the year 1971} &= 548159.652, \\
    \text{Number of persons in the year 1981} &= 683329.097, \\
    \text{Number of persons in the year 1991} &= 846302.688, \\
    \&\text{Number of persons in the year 2001} &= 1027015.247.
\end{align*}

Thus, the formulas / methods are equivalent so far as the estimated values yielded by them are concerned. It is to be noted that the degree of the polynomial is one less than the number of pairs of observations.

The polynomial that represents the given set of numerical data can be used for interpolation at any position of the independent variable lying within its two extreme values. Thus, these formulas / methods can be suitably used in interpolation.

These formulas / methods can also be suitably used in extrapolation and also in inverse interpolation as well as in inverse extrapolation.

Reference


