Arithmetic-Harmonic Mean: Evaluation of Parameter from Observed Data Containing Itself and Random Error

Dhritikesh Chakrabarty

Department of Statistics, Handique Girls’ College, Guwahati, Assam, India
E-mail: dhritikesh.c@rediffmail.com, dhritikeshchakrabarty@gmail.com

Abstract

Some studies have already been made on method of determining the value of parameter from observed data containing the parameter itself and random error due to the reason that the existing statistical methods of estimation in such situation fail in finding out the appropriate value of the parameter. The methods, so developed, suffer from two limitations which are: (i) the methods involve huge computational tasks and (ii) a finite set of observed data may not yield the appropriate value of the parameter in many situations while the number of observations required in the methods may be too large for obtaining the appropriate value of the parameter. For these two limitations one method, based on arithmetic-geometric mean, for the same has recently been developed which involves lesser computational tasks than those involved in the methods developed earlier and which can be applicable in the case of finite set of data. In this paper, another method has been developed for the same which is based on arithmetic-harmonic mean. This paper describes the derivation of the method and one numerical application of the method in determining the central tendency of each of annual maximum and annual minimum of surface air temperature at Guwahati.

Keywords: AHM , parameter, random error, evaluation of parameter, surface air temperature.,central tendency.

1. Introduction

In the situation where the numerical data

\[ x_1, x_2, \ldots, x_N \]
are composed of some parameter $\mu$ and random errors $\varepsilon_i$ which means

$$x_i = \mu + \varepsilon_i, \quad (i = 1, 2, \ldots, N) \quad \text{---------------------- (1.1)}$$


the existing methods of estimation namely least squares method, maximum likelihood method, minimum variance unbiased method, method of moment and method of minimum chi-square, [Aldrich (2000.), Anders (1999), Barnard (1949), Birnbaum (1962), Ivory (1825), Kendall & Stuart (1977), Lehmann & Casella George (1998), Lucien (1990), Walker & Lev (1965)] provides $\frac{1}{N} \sum_{i=1}^{N} x_i$ as estimator of the parameter $\mu$ which suffers from an error $\frac{1}{N} \sum_{i=1}^{N} \varepsilon_i$ [Chakrabarty (2014a, 2014b, 2014c)].

In some recent studies, some methods have been developed for determining the appropriate value of the parameter $\mu$ involved in the model described by equation (1.1) [Chakrabarty (2014a, 2014b, 2014c, 2015a, 2015b, 2015c, 2015d, 2015e, 2015f, 2015g, 2016a, 2016b, 2019a, 2019c)]. In these studies some methods have been developed for determining the appropriate value of the parameter $\mu$ when $\varepsilon_i$ occurs due to random cause.

The first method, developed for the same is based on computing sequence of interval value of $\mu$ with decreasing length of interval and then to find out the shortest interval value of $\mu$ [Chakrabarty (2014a, 2014b, 2014c, 2015d)] while the second one is based on stable mid range and median (Chakrabarty, 2015b) and the third one on the convergence of statistic i.e. some function of the available numerical data (Chakrabarty, 2017a). The fourth one (Chakrabarty, 2018a) has been developed on the basis of Pythagorean means [Kolmogorov (1930), O'Meara (1989), Riedweg (2005), Cornelli, McKirahan & Macris (2013), de Carvalho (2016), Chakrabarty (2018b, 2018c, 2018d, 2018e, 2018f, 2019d, 2019e, 2019f, 2020a)] while the fifth one [Chakrabarty (2016c, 2019b)] for the same is based on the probabilistic convergence of Pythagorean means [Chakrabarty (2017b, 2017b)].

The methods, developed so far, for determining the appropriate value of the parameter from observed data containing the parameter itself and random error involve huge computational tasks. Moreover, a finite set of observed data may not yield the appropriate value of the parameter in many situations while the number of observations required in the methods may be too large for obtaining the appropriate value of the parameter. For these two probable drawbacks, one method for the same purpose was developed which involves lesser computational tasks than those involved in the earlier methods as well as which can
be applicable in the case of finite set of data [Chakrabarty (2019g, 2020b)]. The method developed is based on the concept of Arithmetic-Geometric Mean. In the current attempt, another method has been attempted for the same purpose which is based on the concept of Arithmetic-Geometric Mean. This paper describes the derivation of the method and one numerical application of the method in determining the central tendency of each of annual maximum and annual minimum of surface air temperature at Guwahati.

2. Arithmetic-Harmonic Mean (AHM)

Let \( a_0 \) & \( h_0 \) be respectively the AM (Arithmetic Mean) & HM (Harmonic Mean) of the \( N \) numbers (or values or observations)

\[ x_1, x_2, \ldots, x_N \]

From the inequality of Pythagorean means [Chakrabarty (2016c), Cornelli (2013)] namely

\[ \text{AM} > \text{GM} > \text{HM} \]

(where \( \text{GM} \) means Geometric Mean), it follows that

\[ a_0 > h_0 \]

provided \( x_1, x_2, \ldots, x_N \) are not all equal.

Let \( \{ a_n = a_n(a_0, h_0) \} \) & \( \{ h_n = h_n(a_0, h_0) \} \) be two sequences defined by

\[ a_{n+1} = \frac{1}{2}(a_n + h_n) \]

\[ h_{n+1} = \frac{1}{2}(a_n^{-1} + h_n^{-1}) \]

It is obvious that

\[ a_0 = a_0(a_0, h_0) = a_0 \quad \text{and} \quad h_0 = h_0(a_0, h_0) = h_0 \]

By the inequality of Pythagorean means [Chakrabarty (2016c), Cornelli (2013)],

\[ h_n < a_n \]

and thus

\[ a_{n+1} = \frac{1}{2}(a_n + h_n) \]

\[ \Rightarrow a_{n+1} < \frac{1}{2}(a_n + a_n) \]

\[ \Rightarrow a_{n+1} < a_n \]

This means that the sequence \( \{ a_n = a_n(a_0, h_0) \} \) is non-increasing.

Moreover, the sequence \( \{ a_n = a_n(a_0, h_0) \} \) is bounded below by the smallest of
\[ x_1, x_2, \ldots, x_N \]

(which follows from the fact that both the arithmetic and the harmonic mean of these numbers lie between the smallest and the largest of them).

In the mathematical field of real analysis, the monotone convergence theorem [Weir (1973), Yeh (2006)] states that if a sequence is increasing and bounded above by a supremum, then the sequence will converge to the supremum; in the same way, if a sequence is decreasing and is bounded below by an infimum, it will converge to the infimum.

Thus, by the monotone convergence theorem, the sequence is convergent. Therefore, there exists a finite number \( M_{AH} \) such that,

\[ d'_n \text{ converges to } M_{AH} \text{ as } n \text{ approaches infinity.} \]

Again, \( h'_n \) can be expressed as

\[ h'_n = 2d'_{n+1} - d'_n \]

This implies that the limiting value of \( h'_n \) as \( n \) approaches infinity is \( M_{AH} \).

Therefore,

\[ h'_n \text{ converges to } M_{AH} \text{ as } n \text{ approaches infinity.} \]

Thus, the two sequences \( \{ d'_n = d'_n(a_0, h_0) \} \) & \( \{ h'_n = h'_n(a_0, h_0) \} \) converge to the same point \( M_{AH} \) as \( n \) approaches infinity.

This common converging point \( M_{AH} \) can be termed / named / regarded as the Arithmetic-Harmonic Mean (abbreviated as \( AHM \)) of the \( N \) numbers (or values or observations)

\[ x_1, x_2, \ldots, x_N \]

Accordingly, \( AHM \) can be defined as follows:

**2.1. Definition of Arithmetic-Harmonic Mean (AHM)**

If \( a_0 \) & \( h_0 \) are respectively the AM & the HM of \( N \) numbers (or values or observations) viz \( x_1, x_2, \ldots, x_N \)

Then the two sequences \( \{ d'_n = d'_n(a_0, h_0) \} \) & \( \{ h'_n = h'_n(a_0, h_0) \} \) defined respectively by
\[ d'_{n+1} = \frac{1}{2}(d'_n + h'_n) \quad \text{and} \quad h'_{n+1} = \frac{1}{2}(d'^{-1}_n + h'^{-1}_n) \]\n
where \( d'_0 = d'_0(a_0, h_0) = a_0 \quad \text{and} \quad h'_0 = h'_0(a_0, h_0) = h_0 \) converge to a common limit \( M_{AH} \) which is called the Arithmetic-Harmonic Mean (abbreviated by \( AHM \)) of \( x_1, x_2, \ldots, x_N \) and is denoted here by \( AHM(x_1, x_2, \ldots, x_N) \) i.e.

\[ AHM(x_1, x_2, \ldots, x_N) = M_{AH} \]

**Note:**

1. It is clear that each of \( \{d'_n\} \) & \( \{h'_n\} \) converges to \( M_{AH} \).

   Thus, the converging point of either \( \{d'_n\} \) or \( \{h'_n\} \) can be taken as the value of \( M_{AH} \).

2. Since \( d'_{n+1} \leq d'_n \),

   i.e. \( a_0 \geq d'_1 \geq d'_2 \geq d'_3 \), \ldots

   Therefore, \( a_0 \geq M_{AH} \)

   Again,

   \[ h'_n = 2d'_{n+1} - d'_n \]

   \( \text{and} \quad h'_{n+1} = 2d'_{n+2} - d'_{n+1} \)

   which implies,

   \( h'_n \leq h'_{n+1} \)

   i.e. \( h_0 \leq h'_1 \leq h'_2 \leq h'_3 \), \ldots

   Therefore, \( h_0 \leq M_{AH} \)

   Hence, the following inequality is obtained:

\[ a_0 \geq M_{AH} \geq h_0 \]

This can be stated as follows:

\[ AM \geq AHM \geq HM. \]
3. Evaluation of $\mu$

If the observations

$$x_1, x_2, \ldots, x_N$$

are composed of some parameter $\mu$ and random errors then the observations can be expressed as

$$x_i = \mu + \varepsilon_i, \quad (i = 1, 2, \ldots, N)$$

where

$$\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N$$

are the random errors, which assume positive and negative values in random order, associated to

$$x_1, x_2, \ldots, x_N$$

respectively.

In this case,

$$A(x_1, x_2, \ldots, x_N) \to \mu \quad \text{as} \quad N \to \infty$$

where

$$A(x_1, x_2, \ldots, x_N) = \frac{1}{N} \sum_{i=1}^{N} x_i$$

Again since the observations

$$x_1, x_2, \ldots, x_N$$

consist of $\mu$ and random errors,

therefore, the reciprocals

$$x_1^{-1}, x_2^{-1}, \ldots, x_N^{-1}$$

are composed of $\mu^{-1}$ and random errors different from the respective random errors

$$\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N$$

provided $x_1, x_2, \ldots, x_N$ are all different from zero.

In this case thus

$$x_i^{-1} = \mu^{-1} + \varepsilon_i', \quad (i = 1, 2, \ldots, N)$$

where

$$\varepsilon_1', \varepsilon_2', \ldots, \varepsilon_N'$$
are the random errors, which assume positive and negative values in random order, associated to

\[ x_1^{-1}, x_2^{-1}, \ldots, x_N^{-1} \]

respectively.

In this case,

\[ H(x_1, x_2, \ldots, x_N) \rightarrow \mu \quad \text{as} \quad N \rightarrow \infty \]

where

\[ H(x_1, x_2, \ldots, x_N) = \left( \frac{1}{N} \sum_{i=1}^{N} x_i^{-1} \right)^{-1} \]

This implies that the common converging value of \( A(x_1, x_2, \ldots, x_N) \) and \( H(x_1, x_2, \ldots, x_N) \) is the value of \( \mu \).

It is to be noted that the converging value may not be possible to be obtained for a finite set of observed values namely

\[ x_1, x_2, \ldots, x_N \]

In order to obtain the value of \( \mu \), in this case, let us write

\[ A(x_1, x_2, \ldots, x_N) = A_0 \]
\[ H(x_1, x_2, \ldots, x_N) = H_0 \]

and then define the two interdependent sequences \( \{A_n\} \) and \( \{H_n\} \) as respectively by

\[ A_{n+1} = \frac{1}{2} (A_n + H_n) \]
\[ H_{n+1} = \left\{ \frac{1}{2} \left( A_n^{-1} + H_n^{-1} \right) \right\}^{-1} \]

Then, both of \( A_n \) \& \( H_n \) converges to some real number \( C \) as \( n \) approaches infinity.

Now, it is required to verify whether this \( C \) is equal to \( \mu \).

From the model it is obtained that

\[ A_0 = \mu + \delta_0 \quad \& \quad H_0 = \mu + e_0 \]

The inequality of Pythagorean means namely

\[ AM > HM \]

implies that \( A_0 > H_0 \) i.e. \( \delta_0 > e_0 \)
Thus \( A_1 = \mu + \delta_1 \) where \( \delta_1 = \frac{1}{2} (\delta_0 + e_0) < \delta_0 \)

In general, corresponding to \( A_{n+1} \), it holds that

\[
\delta_{n+1} = \frac{1}{2} (\delta_n + e_n) < \delta_n
\]

This implies, \( \delta_n \) converges to 0 i.e. \( A_n \) converges to \( \mu \).

By the existence of \( AHM \), \( H_n \) also converges to \( \mu \).

Thus, the \( AHM \) of \( x_1, x_2, \ldots, x_N \) is the value of \( \mu \).

4. Application to Numerical Data

Observed data considered here are the data on each of annual maximum & annual minimum of surface air temperature, occurred in temperature periodic year (TPR), at Guwahati during the period from 1969 to 2013. The objective here is to evaluate the central tendency of each of annual maximum & annual minimum of surface air temperature at Guwahati.

4.1. Annual Maximum of Surface Air Temperature at Guwahati:

The following table (Table – 4.1.1) shows the observed data on annual maximum of surface air temperature, occurred in temperature periodic year (TPR), at Guwahati during the period from 1969 to 2013:

<table>
<thead>
<tr>
<th>TPR No (( i ))</th>
<th>Observed Value (( x_i ))</th>
<th>TPR No (( i ))</th>
<th>Observed Value (( x_i ))</th>
<th>TPR No (( i ))</th>
<th>Observed Value (( x_i ))</th>
<th>TPR No (( i ))</th>
<th>Observed Value (( x_i ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>37.1</td>
<td>12</td>
<td>35.1</td>
<td>23</td>
<td>37.4</td>
<td>34</td>
<td>38.0</td>
</tr>
<tr>
<td>2</td>
<td>36.6</td>
<td>13</td>
<td>35.8</td>
<td>24</td>
<td>39.4</td>
<td>35</td>
<td>36.6</td>
</tr>
<tr>
<td>3</td>
<td>36.0</td>
<td>14</td>
<td>36.5</td>
<td>25</td>
<td>36.4</td>
<td>36</td>
<td>38.0</td>
</tr>
<tr>
<td>4</td>
<td>35.7</td>
<td>15</td>
<td>36.7</td>
<td>26</td>
<td>38.1</td>
<td>37</td>
<td>37.3</td>
</tr>
</tbody>
</table>

Table – 4.1.1

Observed Value on Annual Maximum of Surface Air Temperature (in Degree Celsius)
Here the observed values $x_i$ ($i = 1, 2, 3, \ldots \ldots \ldots, 43$) can be assumed to be composed of a parameter $\mu$ (representing the central tendency of annual maximum) and random errors so that the observed values follow the model described by equation (1.1).

**Evaluation of Value of $\mu$ (the central tendency of annual maximum)**

The computed values of the arithmetic mean and the geometric mean of the observed values, shown in Table – 4.1.1, are found to be 37.2093023255814 and 37.175398903562627634836294491501 respectively.

Let us write

$$A_0 = 37.2093023255814 \quad \text{and} \quad H_0 = 37.175398903562627634836294491501$$

In this case the iterations give the values which are given in the following table (Table – 4.1.2):

<table>
<thead>
<tr>
<th>$n$</th>
<th>$A_n$</th>
<th>$H_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>37.2093023255814</td>
<td>37.175398903562627634836294491501</td>
</tr>
<tr>
<td>1</td>
<td>37.192350614572013817418147245751</td>
<td>37.183872827043641055199463981829</td>
</tr>
<tr>
<td>2</td>
<td>37.18811172080782743630880561379</td>
<td>37.188111237636740962336357297681</td>
</tr>
<tr>
<td>3</td>
<td>37.188111479222284199322581455736</td>
<td>37.18811147922228262990772330485</td>
</tr>
<tr>
<td>4</td>
<td>37.188111479222283414615152393111</td>
<td>37.188111479222283022261437861789</td>
</tr>
<tr>
<td>5</td>
<td>37.188111479222283218438295127449</td>
<td>37.188111479222283218438295127449</td>
</tr>
<tr>
<td>6</td>
<td>37.188111479222283218438295127449</td>
<td>37.188111479222283218438295127449</td>
</tr>
</tbody>
</table>
The digits in $A_n$ and $H_n$, which are agreed, have been underlined in the above table.

The $AHM$ of 37.2093023255814 and 37.17539890356262763483629491501 is the common limit of these two sequences which is 37.1881147922283218438295127449.

Thus the value of $\mu$, the central tendency of annual maximum of surface air temperature at Guwahati, obtained by $AHM$, is 37.1881147922283218438295127449 Degree Celsius.

4.2. Annual Minimum of Surface Air Temperature at Guwahati:

The following table (Table – 4.2.1) shows the observed data on annual maximum of surface air temperature, occurred in temperature periodic year (TPR), at Guwahati during the period from 1969 to 2013.

As earlier, the observed values $x_i$ ($i = 1, 2, 3, \ldots \ldots \ldots \ldots , 44$) can in this case also be assumed to be composed of a parameter $\mu$ (representing the central tendency of annual minimum) and random errors so that the observed values follow the model described by equation (1.1).

### Table – 4.2.1

<table>
<thead>
<tr>
<th>TPR No ($i$)</th>
<th>Observed Value ($x_i$)</th>
<th>TPR No ($i$)</th>
<th>Observed Value ($x_i$)</th>
<th>TPR No ($i$)</th>
<th>Observed Value ($x_i$)</th>
<th>TPR No ($i$)</th>
<th>Observed Value ($x_i$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6.6</td>
<td>12</td>
<td>6.4</td>
<td>23</td>
<td>7.4</td>
<td>34</td>
<td>8.0</td>
</tr>
<tr>
<td>2</td>
<td>6.6</td>
<td>13</td>
<td>7.5</td>
<td>24</td>
<td>5.9</td>
<td>35</td>
<td>7.9</td>
</tr>
<tr>
<td>3</td>
<td>5.9</td>
<td>14</td>
<td>8.3</td>
<td>25</td>
<td>8.4</td>
<td>36</td>
<td>6.7</td>
</tr>
<tr>
<td>4</td>
<td>8.2</td>
<td>15</td>
<td>4.9</td>
<td>26</td>
<td>7.8</td>
<td>37</td>
<td>9.6</td>
</tr>
<tr>
<td>5</td>
<td>5.0</td>
<td>16</td>
<td>6.1</td>
<td>27</td>
<td>7.5</td>
<td>38</td>
<td>6.4</td>
</tr>
<tr>
<td>6</td>
<td>6.3</td>
<td>17</td>
<td>7.8</td>
<td>28</td>
<td>9.4</td>
<td>39</td>
<td>7.8</td>
</tr>
<tr>
<td>7</td>
<td>7.4</td>
<td>18</td>
<td>8.6</td>
<td>29</td>
<td>NA</td>
<td>40</td>
<td>9.9</td>
</tr>
</tbody>
</table>
Determination of Value of $\mu$ (the central tendency of annual minimum)

The computed values of arithmetic mean and the harmonic mean of the observed values, shown in Table–4.2.1, are found to be 7.3634146341463414634146341463415 and 7.1543933802823525209849744707569 respectively.

Let us write

\[ A_0 = 7.3634146341463414634146341463415 \quad \text{and} \quad H_0 = 7.1543933802823525209849744707569 \]

In this case the iterations give the values which are given in the following table (Table–4.2.2):

<table>
<thead>
<tr>
<th>$n$</th>
<th>$A_n$</th>
<th>$H_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>7.3634146341463414634146341463415</td>
<td>7.1543933802823525209849744707569</td>
</tr>
<tr>
<td>1</td>
<td>7.2589040072143469921998043085492</td>
<td>7.2573993074510131470508335954916</td>
</tr>
<tr>
<td>2</td>
<td>7.2581516573326800696253189520204</td>
<td>7.2581515793472133602889381001075</td>
</tr>
<tr>
<td>3</td>
<td>7.258151618339946714957128526064</td>
<td>7.2581516183399465054777273757201</td>
</tr>
<tr>
<td>4</td>
<td>7.2581516183399466102174279508921</td>
<td>7.258151618339946610217427950892</td>
</tr>
<tr>
<td>5</td>
<td>7.258151618339946610217427950892</td>
<td>7.258151618339946610217427950892</td>
</tr>
</tbody>
</table>

The digits in $A_n$ and $H_n$, which are agreed, have been underlined in the above table.

The $AHM$ of 7.3634146341463414634146341463415 and 7.1543933802823525209849744707569 is the common limit of these two sequences which is 7.258151618339946610217427950892.
Thus the value of $\mu$, the central tendency of annual minimum of surface air temperature at Guwahati, obtained by $AHM$, is $7.258151618339946610217427950892$ Degree Celsius.

5. Summary of computations

The methods developed so far, for determining the appropriate value of the parameter from observed data containing the parameter itself and random error involve huge computational tasks. The method, based on $AHM$, described here involves lesser computational tasks than those involved in the methods developed so far.

Moreover, a finite set of observed data may not yield the appropriate value of the parameter in many situations while the number of observations required in the methods may be too large for obtaining the appropriate value of the parameter. The method, based on $AHM$, can be applicable in the case of finite set of data.

Regarding the findings obtained on annual maximum and annual minimum of surface air temperature at Guwahati, the following conclusion can be drawn:

5.1. The value of central tendency of annual maximum of surface air temperature at Guwahati, obtained by AHM, is $37.18811147922283218438295127449$ Degree Celsius. However, the value of the same was found to be $37.2$ Degree Celsius by the earlier methods of the same [2014c, 2015a, 2015b, 2015c, 2015d, 2015e, 2017, 2018a, 2019b] while the value for the same obtained by $AGM$ is $37.20079425067069371656824015813$ Degree Celsius (Chakrabarty, 2020b).

5.2. The central tendency of annual maximum of surface air temperature at Guwahati, obtained by $AHM$, is $7.258151618339946610217427950892$ Degree Celsius. However, it was found not possible to obtain the value of central tendency of the same by the earlier methods [2014c, 2015a, 2015b, 2015c, 2015d, 2015e, 2017, 2018a, 2019b] while the value for the same obtained by $AGM$ is $7.0100883193127795038175836675175$ Degree Celsius (Chakrabarty, 2020b).

Reference


